

ON THE BEILINSON–BLOCH–KATO CONJECTURE FOR POLARIZED MOTIVES

HAO PENG

ABSTRACT. We study the Beilinson–Bloch–Kato conjecture for polarized motives. In the conjugate self-dual case, we show that if the central L -value does not vanish, then the associated Bloch–Kato Selmer group with coefficients in a suitable local field vanishes. In the self-dual analytic rank-zero case, we reduce the conjecture to a conjecture in the endoscopic Rankin–Selberg case related to the orthogonal Gross–Prasad periods.

CONTENTS

1. Introduction	1
1.1. The conjugate self-dual case	2
1.2. The self-dual case	4
1.3. Strategy of proof	6
1.4. Notation and conventions	8
1.5. Acknowledgments	10
2. Automorphic representations and Galois representations	10
2.1. The conjugate self-dual case	10
2.2. The self-dual case	11
2.3. Galois theoretic arguments	11
3. The conjugate self-dual Rankin–Selberg case	13
3.1. Unitary Satake parameters and unitary Hecke algebras	13
3.2. Unitary Shimura varieties	15
3.3. Generalized CM type and reflexive closure	16
3.4. Preparation for Tate classes and arithmetic level-raising	16
3.5. Tate classes in the odd rank case	19
3.6. Arithmetic level-raising in the even rank case	21
3.7. First explicit reciprocity law	22
3.8. Admissible places	26
3.9. Proof of Theorem D	27
4. Theta correspondence	31
4.1. The groups	31
4.2. Local Gan–Gross–Prasad conjecture	33
4.3. Local theta lifts and Prasad’s conjectures	34
4.4. Global theta lifts	35
5. Seesaw and proof of main theorems	37
5.1. The conjugate self-dual case	37
5.2. The self-dual case	44
Appendix A. Polarized local Galois representations	49
A.1. Special conjugate self-dual local Galois representations	49
A.2. Special self-dual local Galois representations	53
References	56

1. INTRODUCTION

The Beilinson–Bloch–Kato conjecture for motives vastly generalizes the rank part of the Birch–Swinnerton-Dyer conjecture for elliptic curves. In this article, we study the Beilinson–Bloch–Kato conjecture for motives associated with self-dual (resp. conjugate self-dual) automorphic representations of $\mathrm{GL}_{2r}(\mathbf{A}_F)$, where $F \subset \mathbb{C}$ is a totally real number field (resp. a CM field).

Let $\mathrm{Mot}^{\mathrm{rat}}(F, E)$ denote the pseudo-Abelian category of Chow motives over F with coefficients in a number field E (see, for example, [And04]). For the complex conjugation $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, a polarization of M is an isomorphism $M^c \xrightarrow{\sim} M^\vee(1)$ in the category $\mathrm{Mot}^{\mathrm{rat}}(F, E)$. For any Chow motive $M \in \mathrm{Mot}^{\mathrm{rat}}(F, E)$ and any finite place λ of E , there is a λ -adic realization M_λ , which is a representation of $\mathrm{Gal}(\overline{F}/F)$ with E_λ -coefficients. We consider the Bloch–Kato Selmer group

$$H_f^1(F, M_\lambda) = \ker \left(H^1(F, M_\lambda) \rightarrow \prod_{w \nmid \ell} H^1(I_w, M_\lambda) \times \prod_{w \mid \ell} H^1(F_w, M_\lambda \otimes_{\mathbb{Q}_\ell} \mathbb{B}_{\mathrm{crys}, \ell}) \right),$$

where ℓ is the underlying rational prime of λ and $\mathbb{B}_{\mathrm{crys}, \ell}$ is the ℓ -adic crystalline period ring.

For example, if A is an Abelian variety over F of dimension g and $M = h^{2g-1}(A)(g)$ is the Albanese motive of A with coefficient field $\mathbb{Q}$, then $H_f^1(F, M_\ell)$ is canonically isomorphic to

$$\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_n \mathrm{Sel}_{\ell^n}(A/F)$$

for every rational prime ℓ . Here $\mathrm{Sel}_k(A/F)$ is the mod- k Selmer group of A over F for every positive integer k .

Suppose M is a polarized motive in $\mathrm{Mot}^{\mathrm{rat}}(F, E)$ and λ is a finite place of E . Conjecturally, the L -function $L(s, M_\lambda)$ attached to M_λ has a meromorphic continuation to the entire complex plane, satisfying a functional equation

$$L(s, M_\lambda) = \epsilon(M)c(M)^{-s}L(-s, M_\lambda)$$

where $\epsilon(M) \in \{\pm 1\}$ is the root number and $c(M)$ is the conductor; see [Del79]. Assuming this conjectural functional equation, we recall the following Beilinson–Bloch–Kato conjecture [BK90].

Conjecture (Beilinson–Bloch–Kato).

$$\mathrm{ord}_{s=0} L(s, M_\lambda) = \dim_{E_\lambda} H_f^1(F, M_\lambda) - \dim_{E_\lambda} H^0(F, M_\lambda).$$

We focus on the analytic rank-zero case, that is, when $L(0, M_\lambda)$ is nonzero.

1.1. The conjugate self-dual case. Let $F \subset \mathbb{C}$ be a totally imaginary quadratic extension of a totally real number field $F_+ \subset \mathbb{R}$. We first state a less technical main result.

Theorem A. *Let r be a positive integer and let A be a modular elliptic curve over F_+ . Suppose that $F_+ \neq \mathbb{Q}$ whenever $r > 1$, and that A has no complex multiplication over $\overline{F}$. If the central critical value*

$$L(r, \mathrm{Sym}^{2r-1} A_F)$$

is nonzero, then for all but finitely many rational primes ℓ , the Bloch–Kato Selmer group

$$H_f^1(F, \mathrm{Sym}^{2r-1} H_{\mathrm{\acute{e}t}}^1(A_{\overline{F}}, \mathbb{Q}_\ell)(r))$$

vanishes.

Remark 1.1.1. The finite set of rational primes ℓ that are excluded in Theorem A can be effectively bounded in terms of F , A , and r . The condition $F_+ \neq \mathbb{Q}$ is imposed because Hypothesis 3.2.3 on the cohomology of unitary Shimura varieties is not yet known for $N \geq 4$ if $F_+ = \mathbb{Q}$. This condition is not used elsewhere.

Remark 1.1.2. When $r = 1$, Theorem A recovers part of the Birch–Swinnerton-Dyer conjecture for A_F . If, in addition, $F_+ = \mathbb{Q}$, then this is covered by Kolyvagin’s work [Kol90], which introduced the *Heegner point Euler system*; it uses the Gross–Zagier formula [GZ86] to pass to the analytic rank-one case. For $F_+ \neq \mathbb{Q}$, the corresponding result was later established in [Zha01, Lon06, Lon07, Nek12].

When $r = 1$ and $F_+ = \mathbb{Q}$, there are other approaches to Theorem A. In [Kat04], Kato used p -adic families of Beilinson elements in the K -theory of modular curves to construct Selmer classes via p -adic Hodge theory, known as *Kato’s Euler system*. In [BD05], Bertolini and Darmon developed a different approach that constructs Selmer classes via level-raising congruences on Shimura curves, known as the *bipartite Euler system*.

(see [How06] for a systematic formulation). Under mild additional assumptions, these yield alternative proofs of Kolyvagin’s result that do not invoke the Gross–Zagier formula.

Theorem A is a special case of a more general result concerning the Bloch–Kato Selmer groups of Galois representations attached to conjugate self-dual automorphic representations. We first introduce the automorphic representations we study.

Definition 1.1.3. An isobaric automorphic representation Π of $\mathrm{GL}_N(\mathbf{A}_F)$ with $N \geq 2$ is called a (conjugate self-dual) *relevant automorphic representation* if

- (1) Π is conjugate self-dual in the sense that its contragredient $\Pi^\vee$ is isomorphic to $\Pi \circ \mathbf{c}$;
- (2) for every Archimedean place w of F , Π_w is isomorphic to the irreducible principal series representation induced from the characters $(\arg^{1-N}, \arg^{3-N}, \dots, \arg^{N-1})$, where $\arg : \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ is the argument character defined by the formula $\arg(z) = z/\sqrt{z\bar{z}}$;
- (3) either one of the following holds:
 - (a) Π is cuspidal.
 - (b) N is odd and Π is an isobaric sum of a character of $\mathrm{GL}_1(\mathbf{A}_F)$ with a cuspidal automorphic representation of $\mathrm{GL}_{N-1}(\mathbf{A}_F)$.

A relevant representation Π is called *almost cuspidal* if it is not cuspidal, in which case we write $\Pi = \Pi^\flat \boxplus \chi$, where χ is a conjugate self-dual character of $F^\times \backslash \mathbf{A}_F^\times$.

Remark 1.1.4. Note that our definition of relevant automorphic representations is slightly more general than that of [LTX⁺22, Definition 1.1.3]: A representation Π of $\mathrm{GL}_N(\mathbf{A}_F)$ is relevant in their sense if and only if it is cuspidal and relevant in our sense.

If Π is a relevant automorphic representation, then it is regular algebraic in the sense of [Clo90, Definition 3.12]. Moreover, it is well known that the Asai L -function $L(s, \Pi', \mathrm{As}^{(-1)^N})$ is regular at $s = 1$ for each isobaric factor Π' of Π (see, for example, [FP23, Theorem 9.1]).

We now state our main result in terms of automorphic representations analogous to Theorem A.

Theorem B. *Let r be a positive integer and Π be a relevant automorphic representation of $\mathrm{GL}_{2r}(\mathbf{A}_F)$. Let E be a strong coefficient field of Π (see Definition 3.1.6). If the central critical value*

$$L\left(\frac{1}{2}, \Pi\right)$$

is nonzero, then for every admissible place λ of E with respect to Π , the Bloch–Kato Selmer group

$$H_f^1(F, \rho_{\Pi, \lambda}(r))$$

vanishes. Here $\rho_{\Pi, \lambda}$ is the Galois representation of F with coefficients in E_λ attached to Π as described in Proposition 2.1.1 and Definition 3.1.6.

Remark 1.1.5. In the setting of the unitary Friedberg–Jacquet periods, M. Zanarella studied automorphic representations Π in a framework close to ours, under the additional assumption that Π is self-dual [Zan24]. His argument relies on the conjecture of Leslie–Xiao–Zhang [LXZ25]; see also [LXZ25b] for recent progress on this conjecture.

The notion of admissible places appearing in Theorem B is defined in Definition 5.1.6, which consists of a long list of assumptions. It is expected that all but finitely many finite places are admissible (with respect to Π). Indeed, we have the following family of abstract examples where all but finitely many finite places are admissible.

Theorem C. *Let r and Π be as in Theorem B. Suppose that*

- (1) $F_+ \neq \mathbb{Q}$ if $r > 1$;
- (2) *There exists a finite place w of F such that Π_w is supercuspidal;*
- (3) *There exists a good inert place $\mathfrak{p}$ of F_+ (see Definition 3.3.3) such that $\Pi_{\mathfrak{p}}$ is a Steinberg representation.*

Let E be a strong coefficient field of Π (see Definition 3.1.6). If the central critical value

$$L\left(\frac{1}{2}, \Pi\right)$$

is nonzero, then for all but finitely many finite places λ of E , the Bloch–Kato Selmer group

$$H_f^1(F, \rho_{\Pi, \lambda}(r))$$

vanishes.

Remark 1.1.6. In condition (b) of Theorem C, if F is Galois over $\mathbb{Q}$ or contains an imaginary quadratic field, then a good inert place of F_+ is just a finite place of F_+ that is inert in F .

Using theta correspondence and a Burger–Sarnak type principle for Fourier–Jacobi periods on a pair of global unitary group $(\mathbf{U}_{2r}, \mathbf{U}_{2r})$, we reduce Theorem B to the following theorem concerning central critical values of Rankin–Selberg L -functions. Let $n \geq 2$ be a positive integer. Denote by n_0 and n_1 the unique even and odd numbers in $\{n, n+1\}$, respectively.

Theorem D. *Let Π_0, Π_1 be relevant automorphic representations of $\mathrm{GL}_{n_0}(\mathbf{A}_F)$ and $\mathrm{GL}_{n_1}(\mathbf{A}_F)$, respectively, such that Π_0 is cuspidal and Π_1 is almost cuspidal of the form $\Pi_1 = \Pi_1^\flat \boxplus \mathbf{1}$ where $\mathbf{1}$ is the trivial representation of $\mathrm{GL}_1(\mathbf{A}_F)$. Assume $F_+ \neq \mathbb{Q}$ if $n > 2$, and assume there is a finite place w of F over a place of F_+ inert in F such that $\Pi_{1,w}^\flat$ is square-integrable. Let $E \subset \mathbb{C}$ be a strong coefficient field of both Π_0 and Π_1 (see Definition 3.1.6). If the central critical value*

$$L\left(\frac{1}{2}, \Pi_0\right) \cdot L\left(\frac{1}{2}, \Pi_0 \times \Pi_1^\flat\right)$$

is nonzero, then for every admissible place λ of E with respect to (Π_0, Π_1) , the Bloch–Kato Selmer group

$$H_f^1(F, \rho_{\Pi_0, \lambda}(n_0/2))$$

vanishes.

Remark 1.1.7. Theorem D is analogous to one of the main results of [LTX⁺22] that concerns the analytic rank-zero case, where they assumed that Π_1 is relevant and cuspidal. Via the Gan–Gross–Prasad conjecture [GGP12], which is established in our case in [BPCZ22], the theorem can be regarded as relating nonvanishing unitary Gan–Gross–Prasad periods on a pair of unitary groups $(\mathbf{U}_{2r}, \mathbf{U}_{2r+1})$ to the vanishing of Bloch–Kato Selmer groups.

Remark 1.1.8. The notion of admissible places appearing in Theorem D is defined in Definition 3.8.1, which consists of a long list of assumptions. The admissibility condition here is weaker than the analogous admissibility condition in [LTX⁺22, Definition 8.1.1]. It is expected that if the two automorphic representations Π_0 and Π_1 are not correlated in terms of Langlands functoriality, then all but finitely many finite places of E are admissible with respect to (Π_0, Π_1) . For example, if we assume that

- (1) F is Galois over $\mathbb{Q}$ or contains an imaginary quadratic field,
- (2) for each $\alpha \in \{0, 1\}$, there exists a finite place w_α of F such that Π_{α, w_α} is supercuspidal, and
- (3) there exists a finite place $\mathfrak{p}_+$ of F_+ underlying a unique place $\mathfrak{p}$ of F , such that $\Pi_{0, \mathfrak{p}}$ is a Steinberg representation and $\Pi_{1, \mathfrak{p}}^\flat$ is unramified with Satake parameter not containing 1,

then all but finitely many finite places of E are admissible with respect to (Π_0, Π_1) ; see Lemma 3.8.3.

1.2. The self-dual case. We now state analogous conjectures in the self-dual case. Let $F \subset \mathbb{R}$ be a totally real number field.

Conjecture E. Let r be a positive integer and let A be a modular elliptic curve over F with no complex multiplication over $\overline{F}$. If the central critical value

$$L(r, \mathrm{Sym}^{2r-1} A)$$

is nonzero, then for all rational primes ℓ greater than an effective constant depending only on A and r , the Bloch–Kato Selmer group

$$H_f^1(F, \mathrm{Sym}^{2r-1} H_{\mathrm{\acute{e}t}}^1(A_{\overline{F}}, \mathbb{Q}_\ell)(r))$$

vanishes.

Remark 1.2.1.

- (1) Theorem A is implied by Conjecture E. In fact, we can even drop the assumption $F_+ \neq \mathbb{Q}$ in Theorem A if Conjecture E is true.

- (2) When $r = 1$, Conjecture E is established by Kolyvagin [Kol90] when $F = \mathbb{Q}$ using the Gross–Zagier formula [GZ86], and later generalized to the case when $F \neq \mathbb{Q}$ in [Zha01, Lon06, Lon07, Nek12].
- (3) When $r = 2$ and $F = \mathbb{Q}$, Conjecture E is known by work of H. Wang [Wan22] and N. Sweeting [Swe25] using the bipartite Euler system method.

When $r \cdot [F : \mathbb{Q}]$ is even, it appears that Conjecture E would follow from Theorem A provided the following analytic statement holds:

(NV_r): For any elliptic curve A over F with no complex multiplication over $\overline{F}$ such that $L(r, \text{Sym}^{2r-1} A)$ is nonzero, there exists a totally negative element $D \in F^\times$ effectively bounded by F , A , and r satisfying that the central critical value

$$L(r, \text{Sym}^{2r-1} A^D)$$

is nonzero, where A^D is the twist of A by the quadratic extension $F(\sqrt{D})/F$.

If $r = 1$, (NV₁) holds by the nonvanishing theorem of Friedberg–Hoffstein for quadratic twists with prescribed local behavior; cf. [FH95, Theorem B]. For $r \geq 2$, (NV_r) appears to lie beyond current techniques; even the case $r = 2$ seems difficult (see, for example, [RY23, BFK⁺23, HJL23]).

Alternatively, using theta correspondence and a Burger–Sarnak type principle for Fourier–Jacobi periods on the symplectic-metaplectic pair $(\text{Sp}_{2r}, \widetilde{\text{Sp}}_{2r})$, we show that Conjecture E can be reduced to another conjecture of Gan–Gross–Prasad type, relating nonvanishing of orthogonal Gross–Prasad periods to vanishing of Bloch–Kato Selmer groups.

Conjecture F. Let r be a positive integer and let A be an elliptic curve over F with no complex multiplication over $\overline{F}$. Suppose that there exist

- (1) a self-dual automorphic representation Π of $\text{GL}_{2r+1}(\mathbf{A}_F)$ that is either cuspidal or an isobaric sum of a self-dual cuspidal automorphic representation of $\text{GL}_{2r}(\mathbf{A}_F)$ with a nontrivial quadratic character of $F^\times \backslash \mathbf{A}_F^\times$;
- (2) a pair $(\mathbf{V}, \mathbf{V}_\sharp)$ in which $\mathbf{V}$ is a quadratic space of dimension $2r+1$ over F that is positive definite at every Archimedean place of F satisfying $-\text{disc}(\mathbf{V}) \notin F^{\times 2}$, and $\mathbf{V}_\sharp := \mathbf{V} \oplus Fe$ where e has norm 1;
- (3) cuspidal automorphic representations $\pi_0 \subset \mathcal{A}_0(\text{O}(\mathbf{V}))$ and $\pi_1 \subset \mathcal{A}_0(\text{O}(\mathbf{V}_\sharp))$ with trivial Archimedean components and with Arthur parameters $\text{Sym}^{2r-1} A$ and $\Pi \boxplus \mathbf{1}$, respectively (see Definition 4.4.2);¹ and

- (4) cusp forms $f_0 \in \pi_0$ and $f_1 \in \pi_1$,

such that the orthogonal Gross–Prasad period

$$(1.1) \quad \mathcal{P}_{\text{GP}}(f_0, f_1) := \int_{\text{O}(\mathbf{V})(F) \backslash \text{O}(\mathbf{V})(\mathbf{A}_F)} f_0(h) f_1(\iota(h)) dh$$

is nonzero. Here $\iota : \text{O}(\mathbf{V}) \hookrightarrow \text{O}(\mathbf{V}_\sharp)$ is the embedding induced by the inclusion $\mathbf{V} \subset \mathbf{V}_\sharp$. Let $E \subset \mathbb{C}$ be a strong coefficient field of Π (see Definition 2.2.4). Then there exists an effective constant $N(F, A, r)$ depending only on F , A , and r , such that the Bloch–Kato Selmer group

$$H_f^1(F, \text{Sym}^{2r-1} H_{\text{ét}}^1(A_{\overline{F}}; \mathbb{Q}_\ell)(r))$$

vanishes for all rational primes $\ell > N(F, A, r)$ underlying a preadmissible place λ of E with respect to (A, Π) .

Remark 1.2.2. When $r = 1$ and $F = \mathbb{Q}$, Conjecture F is known by results of Y. Liu [Liu16] under suitable conditions, obtained using Hirzebruch–Zagier cycles and the bipartite Euler system method.

Remark 1.2.3. The notion of preadmissible places appearing in Conjecture F is a preliminary notion defined in Definition 5.2.6. It is expected that, if Π is not correlated to A in the sense of Langlands functoriality, then all but finitely many finite places of E are admissible with respect to (A, Π) . For example, if there exist finite places $\mathfrak{p}, \mathfrak{q}$ of F such that

- (1) A has split multiplicative reduction at $\mathfrak{p}$,
- (2) $\Pi_{\mathfrak{p}}$ is unramified with Satake parameter of the form $\{-1, \alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}\}$ satisfying $\alpha_i \notin \{\pm 1\}$ for every $1 \leq i \leq r$, and

¹Here $\mathbf{1}$ denotes the trivial automorphic character of $\text{GL}_1(\mathbf{A}_F)$.

- (3) Π_q is either supercuspidal or an isobaric sum of a self-dual supercuspidal representation with a quadratic character,

then there exists an effective constant $N(F, A, \Pi_p, \Pi_q)$ depending on F, A, Π_p , and Π_q such that every finite place λ of E with underlying prime ℓ greater than $N(F, A, \Pi_p, \Pi_q)$ is preadmissible with respect to (A, Π) ; see Lemma 5.2.7.

Theorem G. *If Conjecture F holds, then Conjecture E holds.*

Remark 1.2.4. In view of the Gross–Prasad conjecture for orthogonal groups [GP92, GP94, II10], Conjecture F may be viewed as a natural analogue of Theorem D. It will be studied in the author’s forthcoming Ph.D. thesis [Pen26] via orthogonal Shimura varieties and bipartite Euler system method, along the lines of the argument for Theorem D (see §1.3). In particular, for $F = \mathbb{Q}$ we expect to establish Conjecture F, and hence also Conjecture E.

1.3. Strategy of proof. The main innovation of this paper is an extensive use of local and global seesaw identities to deduce Theorem B (resp. Conjecture E) from Theorem D (resp. Conjecture F). The method of seesaw has proved to be a very useful tool in theta lifting of automorphic representations, yet our work seems to be the first to directly apply it to study arithmetic questions.

For simplicity, we restrict to the self-dual case and assume $F = \mathbb{Q}$. Let r be a positive integer and A be an elliptic curve over $\mathbb{Q}$. By Newton–Thorne [NT21], the odd symmetric power $\text{Sym}^{2r-1} A$ is modular and associated with a self-dual cuspidal representation Π_0 of $\text{GL}_{2r}(\mathbb{A}_{\mathbb{Q}})$. Rather than viewing Π_0 as the standard functorial transfer of a cuspidal automorphic representation on a special orthogonal group SO_{2r+1} as in previous work [Liu16, LTX⁺22, Zan24, Swe25], we regard Π_0 as a generic elliptic A -parameter for the metaplectic group $\widetilde{\text{Sp}}_{2r}$ in the Shimura–Waldspurger correspondence framework of [GI18]. In particular, by Arthur’s multiplicity formula proved by Gan–Ichino [GI18], there exists a genuine cuspidal automorphic representation $\tilde{\sigma}_0$ of $\widetilde{\text{Sp}}_{2r}(\mathbb{A}_{\mathbb{Q}})$ with A -parameter Π_0 .² Since the central critical value $L(\frac{1}{2}, \Pi_0)$ is nonzero (and Π_0 is tempered at every rational prime), the Rallis inner product formula [Yam14], together with local conservation relations, yields a positive definite quadratic space $\mathbf{V}_{2r+1}$ of dimension $2r+1$ over $\mathbb{Q}$ such that the global theta lift of $\tilde{\sigma}_0$ to $\text{O}(\mathbf{V}_{2r+1})(\mathbb{A}_{\mathbb{Q}})$ is a nonzero cuspidal automorphic representation π_0 with trivial Archimedean components.

We use the seesaw diagram

$$\begin{array}{ccc} \text{Sp}_{2r} \times \text{Sp}_{2r} & & \text{O}(\mathbf{V}_{2r+2}) \\ | & \searrow & | \\ \text{Sp}_{2r} & & \text{O}(\mathbf{V}_{2r+1}) \times \text{O}(\mathbf{V}_1), \end{array}$$

where $\mathbf{V}_1 = \mathbb{Q}e$ is a 1-dimensional quadratic space with $\|e\| = 1$, and $\mathbf{V}_{2r+2} = \mathbf{V}_{2r+1} \oplus \mathbf{V}_1$. Fix a sufficiently large rational prime ℓ and a nontrivial additive character ψ of $\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}$. Suppose we can find a cuspidal automorphic representation σ_1 of $\text{Sp}_{2r}(\mathbb{A}_{\mathbb{Q}})$ such that the *Fourier–Jacobi period integral*

$$\mathcal{FJ}(\tilde{\varphi}_0, \varphi_1; \phi) := \int_{\text{Sp}_{2r}(\mathbb{Q}) \backslash \text{Sp}_{2r}(\mathbb{A}_{\mathbb{Q}})} \tilde{\varphi}_0(g) \varphi_1(g) \theta(g; \phi) dg$$

is nonzero on the pair $(\sigma_1, \tilde{\sigma}_0)$ for some Schwartz function ϕ , where $\theta(g; \phi)$ is the theta function. Then it follows from the global seesaw identity that the theta lift of σ_1 to $\text{O}(\mathbf{V}_{2r+2})$ is a nonzero cuspidal automorphic representation π_1 , and the orthogonal Gross–Prasad period integral (1.1) is nonzero on the pair (π_0, π_1) . If we can further guarantee that

- (1) π_1 has trivial Archimedean component,
- (2) the Arthur parameter of π_1 is of the form $\Pi \boxplus \mathbf{1}$ as in the statement of Conjecture F; and
- (3) ℓ underlies a preadmissible place λ of E with respect to (A, Π) ,

then Conjecture E follows.

The shape of π_1 is determined by the shape of σ_1 via Prasad’s conjecture [AG17]. Fix a large prime p . If the L -parameter of $\sigma_{1,p}$ contains a chosen $2r$ -dimensional irreducible local Galois representation ϕ_p as a

²In fact, we twist Π_0 by a nontrivial quadratic character so that the quadratic space $\mathbf{V}_{2r+1}$ (defined below) satisfies $-\text{disc}(\mathbf{V}_{2r+1}) \notin \mathbb{Q}^{\times 2}$.

subrepresentation, we would know Π is either cuspidal or an isobaric sum of a self-dual cuspidal automorphic representation and a quadratic Dirichlet character. Note that we cannot guarantee that Π is cuspidal, because there exist no irreducible self-dual local Galois representations of odd dimension greater than one (when p is odd); see [Pra99, Proposition 4]. Condition (1) and (3) would follow if we can control the Archimedean place of σ_1 and can choose ϕ_p with desired good properties.

To achieve these requirements, we prove a Burger–Sarnak type principle for Fourier–Jacobi periods on the pair $(\mathrm{Sp}_{2r}, \widetilde{\mathrm{Sp}}_{2r})$, in the spirit of [BS91, HL98, Pra07, Zha14]. More precisely, suppose

- (1) $\sigma_{1,p}$ is a supercuspidal representation of $\mathrm{Sp}_{2r}(\mathbb{Q}_p)$ that is induced from a compact open subgroup such that the pair $(\tilde{\sigma}_{0,p}, \sigma_{1,p})$ satisfies the Fourier–Jacobi case of the local Gan–Gross–Prasad restriction problem:

$$(1.2) \quad \mathrm{Hom}_{\mathrm{Sp}_{2r}(\mathbb{Q}_p)}(\tilde{\sigma}_{0,p} \otimes \omega_{\psi_p} \otimes \sigma_{1,p}, \mathbb{C}) \neq 0,$$

where ω_{ψ_p} is the local Weil representation associated to ψ_p .

- (2) The contragredient of $\sigma_{1,\infty}$ is a holomorphic discrete series of $\mathrm{Sp}_{2r}(\mathbb{R})$ with scalar lowest K -type of weight $(r+1, \dots, r+1)$.

We show that there exists a cuspidal automorphic representation σ_1 of Sp_{2r} which globalizes $\sigma_{1,p}$ and $\sigma_{1,\infty}$ simultaneously, such that the Fourier–Jacobi period integral on the pair (σ_1, σ_0) is nonzero. The local restriction condition (1.2) then follows from the (now established) local Gan–Gross–Prasad conjecture, Prasad’s conjecture, and a local seesaw identity; see §4.

The local Galois representation ϕ_p used to enforce the pre-admissibility condition at some place λ above ℓ will be constructed in Appendix A. Let $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ be a fixed isomorphism which induces a place λ of E . We require ϕ_p to satisfy:

- (1) ϕ_p is self-dual of orthogonal type;
- (2) ϕ_p is residually absolutely irreducible;
- (3) there exists an arithmetic Frobenius lift $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ such that the eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ of $\iota_\ell \rho(\mathrm{Frob}_p)$ are ℓ -adic units and their reductions in $\mathbb{F}_\ell$ avoid a prescribed finite subset of $\mathbb{F}_\ell$; moreover, $\alpha_i^2 \neq p^2 \alpha_j^2$ in $\mathbb{F}_\ell$ for any $1 \leq i \neq j \leq 2r$.

The explicit construction is more complicated than we expected. Indeed, in the conjugate self-dual variant, we need to split and distribute the analogous requirements between two distinct finite places.

We now turn to the conjugate self-dual setting and discuss the proof of Theorem D, which is another main theorem. Following the bipartite Euler system arguments via level-raising congruences, pioneered by Bertolini and Darmon for Shimura curves [BD05], we bound the Bloch–Kato Selmer group by constructing global Galois cohomology classes that are deeply ramified at prescribed primes. These classes originate from the cohomology of products of unitary Shimura varieties attached to (standard indefinite) unitary groups $\mathbf{U}_{n_0}$ and $\mathbf{U}_{n_1}$ via level-raising congruences, and are realized as the image of the diagonal cycle under the Hecke-localized Abel–Jacobi map. Their ramifications are detected by relating them to unitary Gan–Gross–Prasad periods on definite Shimura sets through the basic uniformization of the special fibers of the integral models—this is the so-called *first explicit reciprocity law*.

Our argument follows [LTX+22] but requires modifications for the almost cuspidal setting. The results of [LTX+22] do not apply verbatim, since several of their standing hypotheses are tailored to the cuspidal case. For example, the computation of the Hecke–Galois module of the Shimura varieties is more delicate: when π is a cuspidal representation of $\mathbf{U}_{n_1}$ with base change $\mathrm{BC}(\pi) \cong \Pi_1 = \Pi_1^p \boxplus \mathbf{1}$, then the π^∞ -isotypic part of the middle-degree (projective limit) cohomology of the Shimura variety

$$H_{\mathrm{ét}}^{n_1-1} \left(\mathrm{Sh}(\mathbf{U}_{n_1})_{\overline{F}}, \overline{\mathbb{Q}_\ell} \left(\frac{n_1-1}{2} \right) \right) [\iota_\ell \pi^\infty]$$

is a $\overline{\mathbb{Q}_\ell}[\mathrm{Gal}(\overline{F}/F)]$ -module isomorphic to either the trivial character or the Galois representation $\rho_{\Pi_1^p, \lambda}^c(r)$, determined by Arthur’s multiplicity formula. Here $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ is a fixed isomorphism inducing a place λ of E . More subtly, our construction of Π_1 via the Burger–Sarnak type principle fixes its local components at finitely many places, but does not a priori control ramifications at the remaining places. To compensate, we replace the notion of admissibility of [LTX+22] with a weaker variant adapted to the analytic rank-zero situation.

As in [LTX⁺22], the geometric input has two parts: (i) the study of Tate cycles in the special fiber of the semistable integral model of Shimura varieties attached to $\mathbf{U}_{n_1}$; and (ii) an arithmetic level-raising property for Shimura varieties attached to $\mathbf{U}_{n_0}$. In our application, non-cuspidality is allowed only on the odd-unitary side, while the even-unitary representation remains cuspidal. Moreover, the Tate cycle argument works provided the Satake parameter at the given unramified place is “generic” enough.

Finally, under the hypotheses of Theorem D, we are not able to prove the vanishing of the larger Bloch–Kato Selmer group

$$H_f^1(F, \rho_{\Pi_0, \lambda} \otimes \rho_{\Pi_1, \lambda}(n)),$$

although this is predicted by the Beilinson–Bloch–Kato conjecture. This limitation is intrinsic to our simplifying conditions when applying the bipartite Euler system method: Let $R^{\mathfrak{p}}$ be a self-dual lattice in $\rho_{\Pi_0, \lambda} \otimes \rho_{\Pi_1, \lambda}(n)$. For any very good inert place $\mathfrak{p}$ of F_+ at which both arithmetic level-raising and the Tate cycle conditions apply, we cannot show deep ramification of the Hecke-localized Abel–Jacobi image of the diagonal cycle in the singular part of the local Galois cohomology

$$H_{\text{sing}}^1(F_{\mathfrak{p}}, R^{\mathfrak{p}}/\lambda^m),$$

for any $m \geq 1$. Indeed, under further conditions we impose, these cohomology spaces vanish; see §3.9. We do not know how to circumvent this limitation.

Let us briefly summarize this article. In §2, we recall certain background materials related to automorphic representations and Galois representations. In §3, we consider the conjugate self-dual Rankin–Selberg case. In §§3.1–3.4, we collect certain background results from [LTX⁺22] and extend them to the almost cuspidal situation. In §§3.5–3.7, we compute the local part of the Abel–Jacobi image of the diagonal cycle. In §3.8, we define the notion of admissible places in the almost cuspidal situation, and check in good situations that all but finitely many finite places are admissible. In §3.9, we prove Theorem D. In §4, we collect the necessary background results related to theta correspondence that will be used in §5. Finally, in §5, we apply the Burger–Sarnak type principle and seesaw relation to prove the main theorems: Theorems A, B and C are proved in §5.1, and Theorem G is proved in §5.2. In Appendix A, we construct certain (conjugate) self-dual local Galois representations with good properties, which will be used in the Burger–Sarnak type principle for Fourier–Jacobi periods.

1.4. Notation and conventions. In this subsection, we set up some common notations and conventions for the entire article, including the appendix.

Notation 1.4.1 (Generalities).

- Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ be the monoid of nonnegative integers and set $\mathbb{Z}_+ = \mathbb{N} \setminus \{0\}$. We write $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the integers, rational numbers, real numbers, and complex numbers, respectively.
- We take square roots only of positive real numbers and always choose the positive root.
- For any set S , we denote by $\mathbf{1}_S$ the characteristic function of S , and by $\text{id}_S : S \rightarrow S$ the identity map. We write id for id_S if S is clear from context. Let $\#S$ be the cardinality of S .
- For any set X , let $\mathbf{1} \in X$ denote the distinguished trivial element (this notation is only used when the notion of triviality is clear from context).
- The eigenvalues or generalized eigenvalues of a matrix over a field k are counted with multiplicity, i.e., by the dimension of the corresponding eigenspace or generalized eigenspace.
- For each rational prime p , we fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ with residue field $\overline{\mathbb{F}_p}$. For every integer $r \in \mathbb{Z}_+$, we denote by $\mathbb{Q}_{p^r}$ the unique unramified extension of $\mathbb{Q}_p$ of degree r inside $\overline{\mathbb{Q}_p}$, and by $\mathbb{F}_{p^r}$ its residue field.
- We use standard notations from category theory. The category of sets is denoted by $\mathbf{Set}$. The category of schemes is denoted by $\mathbf{Sch}$.
- All rings are commutative and unital, and ring homomorphisms preserve units.
- If a base ring is not specified in the tensor operation $\otimes$, then it is $\mathbb{Z}$.
- For a ring L and a set S , denote by $L[S]$ the L -module of L -valued functions on S of finite support.
- For each square matrix M over a ring, we write $M^{\top}$ for its transpose.
- Suppose $\tilde{\Gamma}, G$ are groups, $\Gamma \subset \tilde{\Gamma}$ is a subgroup, and L is a ring.

- We denote by Γ^{ab} the maximal abelian quotient of Γ ;
- For a homomorphism $\rho : \Gamma \rightarrow \text{GL}_r(L)$ for some $r \in \mathbb{Z}_+$, we denote by $\rho^\vee : \Gamma \rightarrow \text{GL}_r(L)$ the contragredient homomorphism, which is defined by the formula $\rho^\vee(x) = (\rho(x)^\top)^{-1}$.
- For a group homomorphism $\rho : \Gamma \rightarrow G$ and an element $\gamma \in \tilde{\Gamma}$ that normalizes Γ , let $\rho^\gamma : \Gamma \rightarrow G$ denote the homomorphism defined by $\rho^\gamma(x) = \rho(\gamma x \gamma^{-1})$.
- We say that two homomorphisms $\rho_1, \rho_2 : \Gamma \rightarrow G$ are conjugate if there exists an element $g \in G$ such that $\rho_1 = g \circ \rho_2 \circ g^{-1}$.
- For any positive integer $n \in \mathbb{Z}_+$, let μ_n denote the finite diagonalizable group scheme over $\mathbb{Z}$ of n -th roots of unity.
- Denote by $\mathbf{c} \in \text{Gal}(\mathbb{C}/\mathbb{R})$ the complex conjugation.
- For each field k , we denote by $\text{char } k$ the characteristic of k .
- If G is a real Lie group or a totally disconnected locally compact group and π is an irreducible admissible representation of G , we denote by $\pi^\vee$ the contragredient of π . We do not use $\tilde{\pi}$ for the contragredient of π .

Notation 1.4.2 (Number fields). A subfield of $\mathbb{C}$ is called a number field if it is a finite extension of $\mathbb{Q}$. Suppose F is a number field.

- We denote by $\mathcal{O}_F$ the ring of integers of F . We will not distinguish between prime ideals of $\mathcal{O}_F$ and the corresponding finite places of F ; we denote by Σ_F^{fin} the set of finite places of F , by $\Sigma_F^\infty = \text{Hom}(F, \mathbb{C})$ the set of infinite places (also called Archimedean places) of F , and by $\Sigma_F = \Sigma_F^{\text{fin}} \cup \Sigma_F^\infty$ the set of all places of F .
- For each finite set Σ of finite places of F , we write

$$\mathbf{A}_{F, \Sigma} := \prod_{v \in \Sigma} F_v, \quad \mathbf{A}_F^\Sigma := \prod'_{v \in \Sigma_F \setminus \Sigma} F_v, \quad \mathbf{A}_F := \mathbf{A}_F^\infty \times (F \otimes_{\mathbb{Q}} \mathbb{R})$$

If $F = \mathbb{Q}$, we omit $\mathbb{Q}$ from the notation.

- Let $\overline{F}$ denote the Galois closure of F in $\mathbb{C}$, and set $\text{Gal}_F = \text{Gal}(\overline{F}/F)$.
- For each rational prime ℓ , let $\varepsilon_\ell : \text{Gal}_F \rightarrow \mathbb{Z}_\ell^\times$ denote the ℓ -adic cyclotomic character. If v is a finite place of F , we continue to write ε_ℓ for its restriction to Gal_{F_v} .
- We fix the following conventions. For each finite place $v \in \Sigma_F^{\text{fin}}$:
 - write $\mathcal{O}_v$ and F_v for the completion of $\mathcal{O}_F$ (resp. F) at v ;
 - let κ_v denote the residue field, $\|v\| := \#\kappa_v$, and write $\text{char } \kappa_v$ for the residue characteristic.
 - we fix an algebraic closure $\overline{F}_v$ of F_v and an embedding $\iota_v : \overline{F} \hookrightarrow \overline{F}_v$ extending $F \hookrightarrow F_v$; via ι_v we regard $\text{Gal}_{F_v} := \text{Gal}(\overline{F}_v/F_v)$ as a decomposition subgroup of Gal_F ;
 - for any map $r : \text{Gal}_F \rightarrow X$, we write $r_v := r|_{\text{Gal}_{F_v}}$;
 - let $I_v \subset \text{Gal}_{F_v}$ denote the inertia subgroup;
 - fix an algebraic closure $\overline{\kappa}_v$ of κ_v , and identify $\text{Gal}_{\kappa_v} := \text{Gal}(\overline{\kappa}_v/\kappa_v)$ with Gal_{F_v}/I_v ,
 - fix $\phi_v \in \text{Gal}_{F_v}$ lifting the arithmetic Frobenius in Gal_{κ_v} , and
 - let W_{F_v} denote the Weil group, and denote by $\text{Art}_v : F_v^\times \rightarrow W_{F_v}^{\text{ab}}$ the local reciprocity map (also called the Artin map), normalized so that uniformizers are sent to geometric Frobenius classes.
 - for every automorphism $\tau \in \text{Aut}(F)$, denote by v^τ the place defined by $v^\tau(x) := v(\tau^{-1}x)$ for every $x \in F$.
- For each finite set S of rational primes, set $\Sigma_F(S) := \{v \in \Sigma_F^{\text{fin}} : \text{char } \kappa_v \in S\}$. If $S = \{p\}$ is a singleton, we write simply $\Sigma_F(p) := \Sigma_F(\{p\}) = \{v \in \Sigma_F^{\text{fin}} : v|p\}$.
- Two subsets Σ_1, Σ_2 of finite places of F are called *strongly disjoint* if $\{\text{char } \kappa_v : v \in \Sigma_1\}$ is disjoint from $\{\text{char } \kappa_v : v \in \Sigma_2\}$.

Notation 1.4.3 (Automorphic representations). Suppose F is a number field. Let G be either the metaplectic double cover $\widetilde{\text{Sp}}_{2n}$ of a symplectic group Sp_{2n} over F or an algebraic group over F whose central connected component is a connected reductive group.

- If G is a metaplectic group $\widetilde{\text{Sp}}_{2n}$, then an automorphic form f on $G(\mathbf{A}_F)$ is called *genuine* if the nontrivial element in $\ker(\widetilde{\text{Sp}}_{2n}(\mathbf{A}_F) \rightarrow \text{Sp}_{2n}(\mathbf{A}_F))$ acts by -1 on f . For simplicity, if G is not

metaplectic, then every automorphic form on $G(\mathbf{A}_F)$ is called genuine. We denote by $\mathcal{A}_0(G(\mathbf{A}_F))$ the space of genuine cusp forms on $G(\mathbf{A}_F)$.

- Suppose π is an automorphic representation of $G(\mathbf{A}_F)$.
 - We write π_v for its local component at v , for every place v of F .
 - We denote by $\pi^\vee$ the contragredient of π .
 - For any automorphism $\tau \in \text{Aut}(F)$, we denote by π^τ the automorphic representation of $G(\mathbf{A}_F)$ satisfying $\pi_v^\tau \cong \pi_{v^\tau}$ for every place v of F .
 - For any cuspidal genuine automorphic representation $\pi \subset \mathcal{A}_0(G(\mathbf{A}_F))$, we write $\bar{\pi}$ for its conjugation.

1.5. Acknowledgments. I wish to thank Rui Chen and Jialiang Zou for many valuable discussions on the theta correspondence. I am grateful to Yifeng Liu for sharing an earlier draft of [LTX+25]. I also thank Weixiao Lu, Hang Xue, and Murilo C. Zanarella for helpful conversations, and Daniel Disegni, Zhiyu Zhang for comments on an earlier draft. Finally, I am deeply indebted to my Ph.D. advisor, Wei Zhang, for his invaluable guidance and encouragement.

2. AUTOMORPHIC REPRESENTATIONS AND GALOIS REPRESENTATIONS

In this section, we introduce the automorphic representations relevant to us and their associated Galois representations.

2.1. The conjugate self-dual case. In this subsection, we fix a positive integer $N \in \mathbb{Z}_+$, an imaginary quadratic extension F of a totally real number field F_+ , and a relevant representation Π of $\text{GL}_N(\mathbf{A}_F)$ (see Definition 1.1.3).

If $\mathbf{V}$ is a Hermitian space of dimension N over F and π is a discrete automorphic representation of $\text{U}(\mathbf{V})(\mathbf{A}_{F_+})$, let $\text{BC}(\pi)$ denote the automorphic base change of π as defined in [LTX+22, Definition 3.2.3] (see also Definition 4.4.2), which always exists by [CZ24, Theorem 2.1].

Proposition 2.1.1.

- (1) For every finite place w of F , Π_w is tempered.
- (2) Suppose Π is cuspidal. For every rational prime ℓ and every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, there exists a semisimple continuous homomorphism

$$\rho_{\Pi, \iota_\ell} : \text{Gal}_F \rightarrow \text{GL}_N(\overline{\mathbb{Q}_\ell}),$$

unique up to conjugation, satisfying that

$$\text{WD}_\ell(\rho_{\Pi, \iota_\ell}|_{\text{Gal}_{F_w}})^{\text{F-ss}} \cong \iota_\ell \text{rec}_N \left(\Pi_w \otimes |\det|^{\frac{1-N}{2}} \right),$$

for every finite place w of F , where rec_N is the local Langlands correspondence for $\text{GL}_N(F_w)$. Moreover, $\rho_{\Pi, \iota_\ell}^\vee$ and $\rho_{\Pi, \iota_\ell}^\vee(1-N)$ are conjugate.

- (3) Suppose N is odd and $\Pi = \Pi^\flat \boxplus \chi$ is almost cuspidal. For every rational prime ℓ and every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, there exist semisimple continuous homomorphisms

$$\rho_{\Pi^\flat, \iota_\ell} : \text{Gal}_F \rightarrow \text{GL}_{N-1}(\overline{\mathbb{Q}_\ell}), \quad \rho_{\chi, \iota_\ell} : \text{Gal}_F \rightarrow \text{GL}_1(\overline{\mathbb{Q}_\ell}),$$

unique up to conjugation, satisfying that

$$\text{WD}_\ell(\rho_{\Pi^\flat, \iota_\ell}|_{\text{Gal}_{F_w}})^{\text{F-ss}} \cong \iota_\ell \text{rec}_N \left(\Pi_w^\flat \otimes |\det|^{\frac{1-N}{2}} \right)$$

and

$$\text{WD}_\ell(\rho_{\chi, \iota_\ell}|_{\text{Gal}_{F_w}})^{\text{F-ss}} = \iota_\ell \left(\chi_w \otimes |\det|^{\frac{1-N}{2}} \right) \circ \text{Art}_w^{-1},$$

for every finite place w of F , where rec_{N-1} is the local Langlands correspondence for $\text{GL}_{N-1}(F_w)$. Moreover, $\rho_{\Pi^\flat, \iota_\ell}$ and $\rho_{\Pi^\flat, \iota_\ell}^\vee(1-N)$ are conjugate. Let ρ_Π denote the direct sum Galois representation $\rho_{\Pi^\flat, \iota_\ell} \boxplus \rho_{\chi, \iota_\ell}$.

Proof. These follow from standard results, see for example [CH13, Theorem 3.2.3], [Car12, Theorem 1.1] and [Car14, Theorem 1.1] $\square$

Remark 2.1.2. If χ is trivial, then ρ_{χ, ι_ℓ} equals $\varepsilon_\ell^{(1-N)/2}$.

Lemma 2.1.3. *Let ℓ be a rational prime with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. If N is odd, then $\rho_{\Pi, \iota_\ell}(\frac{N-1}{2})$ is pure of weight 0 at every finite place w of F . If N is even, then $\rho_{\Pi, \iota_\ell}(\frac{N}{2})$ is pure of weight -1 at every finite place w of F .*

Proof. It suffices to show that $\rho_{\Pi, \iota_\ell}(\frac{N-1}{2})$ (resp. $\rho_{\Pi, \iota_\ell}(\frac{N}{2})$) is pure of some weight when N is odd (resp. even). By [TY07, Lemma 1.4(3)] and Proposition 2.1.1, this follows from the fact that Π_w is tempered for any finite place w of F . $\square$

2.2. The self-dual case. In this subsection, we fix a positive integer r and a totally real number field F . Let Σ^{bad} denote the (finite) set of finite places of F whose underlying rational prime ramifies in F .

Definition 2.2.1. An isobaric automorphic representation Π of $\text{GL}_{2r+1}(\mathbf{A}_F)$ is called a (self-dual) *relevant automorphic representation* if

- (1) Π is self-dual in the sense that its contragredient $\Pi^\vee$ is isomorphic to Π ;
- (2) Π has nontrivial central character $\chi_{(-1)^{r+1}\mathfrak{d}}$, where $\chi_{(-1)^{r+1}\mathfrak{d}}$ is the quadratic character of $\mathbf{A}_F$ attached to a quadratic extension $F(\sqrt{(-1)^{r+1}\mathfrak{d}})$ of F , where $\mathfrak{d}$ is a totally positive element in $F^\times$;
- (3) Π_∞ has infinitesimal character $(r-1, r-2, \dots, 1-r)$; and
- (4) Π is either cuspidal or an isobaric sum of a cuspidal automorphic representation of $\text{GL}_{2r}(\mathbf{A}_F)$ and a nontrivial quadratic character of $F^\times \backslash \mathbf{A}_F^\times$.

We fix a relevant representation Π of $\text{GL}_{2r+1}(\mathbf{A}_F)$, and denote by Σ^Π the smallest (finite) set of finite places of F containing Σ^{bad} such that Π_v is unramified for every finite place v of F not in Σ^Π .

Proposition 2.2.2.

- (1) *For every finite place v of F , Π_v is tempered.*
- (2) *For every rational prime ℓ and every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, there exists a semisimple continuous homomorphism*

$$\rho_{\Pi, \iota_\ell} : \text{Gal}_F \rightarrow \text{GL}_{2r+1}(\overline{\mathbb{Q}_\ell}),$$

unique up to conjugation, satisfying that

$$\text{WD}_\ell(\rho_{\Pi, \iota_\ell}|_{\text{Gal}_{F_w}})^{\text{F-ss}} \cong \iota_\ell \text{rec}_{2r+1}(\Pi_w \otimes |\det|^{-r}),$$

for every finite place v of F , where rec_{2r+1} is the local Langlands correspondence for $\text{GL}_{2r+1}(F_v)$. Moreover, ρ_{Π, ι_ℓ} and $\rho_{\Pi, \iota_\ell}^\vee(-2r)$ are conjugate.

Proof. These follow from standard results, see for example [CH13, Theorem 3.2.3], [Car12, Theorem 1.1] and [Car14, Theorem 1.1] $\square$

Definition 2.2.3. For each finite place w of F not lying above Σ_+^Π , let $\alpha(\Pi_w)$ denote the Satake parameter of Π_w , and let $\mathbb{Q}(\Pi_w)$ denote the subfield of $\mathbb{C}$ generated by the coefficients of the polynomial

$$\prod_{\alpha \in \alpha(\Pi_w)} (T - \alpha) \in \mathbb{C}[T].$$

We define the *coefficient field* (also called the *Hecke field*) of Π to be the compositum of the fields $\mathbb{Q}(\Pi_w)$ for all finite places w of F not lying above Σ_+^Π , denoted by $\mathbb{Q}(\Pi)$.

Definition 2.2.4. We say a number field $E \subset \mathbb{C}$ is a *strong coefficient field* of Π if E contains $\mathbb{Q}(\Pi)$, and for every finite place λ of E with underlying prime ℓ , there exists a continuous homomorphism

$$\rho_{\Pi, \lambda} : \text{Gal}_F \rightarrow \text{GL}_{2r+1}(E_\lambda)$$

necessarily unique up to conjugation, such that for every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ inducing the place λ , $\rho_{\Pi, \lambda} \otimes_{E_\lambda} \overline{\mathbb{Q}_\ell}$ and ρ_{Π, ι_ℓ} (see Proposition 2.2.2) are conjugate.

Remark 2.2.5. By the argument of [CH13, Proposition 3.2.5], a strong coefficient field of Π exists.

2.3. Galois theoretic arguments. Let F_+ be a subfield of $\mathbb{R}$ and F be a quadratic extension of F_+ contained in $\mathbb{C}$ that is not contained in $\mathbb{R}$. We fix an odd rational prime ℓ that is unramified in F , and consider a finite extension $E_\lambda/\mathbb{Q}_\ell$, with ring of integers $\mathcal{O}_\lambda$ and the maximal ideal λ of $\mathcal{O}_\lambda$. We freely use the notation of [LTX⁺22, §2]. For example,

- If Γ is a topological group and L is a $\mathbb{Z}_\ell$ -ring that is finite over either $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, then an $L[\Gamma]$ -module M is called *weakly semisimple* if M is an object of $\text{Mod}(\Gamma, L)$, and the natural map $M^\Gamma \rightarrow M_\Gamma$ is an isomorphism.
- For each positive integer $N \in \mathbb{Z}_+$, we define the group scheme $\mathcal{G}_N := (\text{GL}_N \times \text{GL}_1) \rtimes \{1, \mathbf{c}\}$ with $\mathbf{c}^2 = 1$ and $\mathbf{c}(g, \mu)\mathbf{c} = (\mu g^{-\top}, \mu)$ for $(g, \mu) \in \text{GL}_N \times \text{GL}_1$. Denote by $\nu : \mathcal{G}_N \rightarrow \text{GL}_1$ the homomorphism such that $\nu|_{\text{GL}_N \times \text{GL}_1}$ is the projection to the GL_1 factor and $\nu(\mathbf{c}) = -1$.
- For an $\mathcal{O}_\lambda$ -module M and an element $x \in M$, the *exponent* of x is defined to be

$$\exp_\lambda(x, M) := \min\{d \in \mathbb{N} \cup \{\infty\} \mid \lambda^d x = 0\}.$$

- For a finite place w of F over ℓ and an object R in $\text{Mod}(F_w, \mathcal{O}_\lambda)$ that is crystalline with Hodge–Tate weights in $[a, b]$ where b and $-a$ are nonnegative integers and $b - a \leq \ell - 2$, let $H_{\text{ns}}^1(F_w, R)$ denote the $\mathcal{O}_\lambda$ -submodule of $H^1(F_w, R)$ consisting of elements s represented by an extension

$$0 \rightarrow R_0 \rightarrow R_s \rightarrow \mathbb{Z}_\ell \rightarrow 0$$

in the category $\text{Mod}(F_w, \mathbb{Z}_\ell)$ such that R_s is crystalline. Here R_0 is the underlying $\mathbb{Z}_\ell[\text{Gal}_{F_w}]$ -module of R .

- For a finite place w of F not over ℓ and an object R in $\text{Mod}(F_w, \mathcal{O}_\lambda)$, we set $H_{\text{sing}}^1(F_w, R) := H^1(I_{F_w}, R)^{\text{Gal}_{\kappa_w}}$, and denote by $H_{\text{ns}}^1(F_w, R)$ the kernel of the canonical map

$$\partial_w : H^1(F_w, R) \rightarrow H_{\text{sing}}^1(F_w, R).$$

$H_{\text{ns}}^1(F_w, R)$ is canonically isomorphic to $H^1(\kappa_w, R^{I_{F_w}})$.

We recall the following definition of Bloch–Kato Selmer groups from [BK90].

Definition 2.3.1. For an object $R \in \text{Mod}(F, E_\lambda)$, the *Bloch–Kato Selmer group* $H_f^1(F, R)$ attached to R is defined to be

$$H_f^1(F, R) := \ker \left(H^1(F, R) \rightarrow \prod_{w \in \Sigma_F^{\text{fin}} \setminus \Sigma_F(p)} H_{\text{sing}}^1(F_w, R) \times \prod_{w \in \Sigma_F(p)} H^1(F_w, R \otimes_{\mathbb{Q}_\ell} \mathbb{B}_{\text{crys}}) \right)$$

Definition 2.3.2. For an object $R \in \text{Mod}(F, \mathcal{O}_L)_{\text{fr}}$, the (integral) *Bloch–Kato Selmer group* $H_f^1(F, R)$ attached to R is defined to be the inverse image of $H_f^1(F, R \otimes \mathbb{Q})$ under the natural map

$$H^1(F, R) \rightarrow H^1(F, R \otimes \mathbb{Q}).$$

Moreover, for each $m \in \mathbb{Z}_+ \cup \{\infty\}$, the (mod- λ^m) *Bloch–Kato Selmer group* $H_{f,R}^1(F, \overline{R}^{(m)})$ is defined to be the image of $H_f^1(F, R)$ under the natural map $H^1(F, R) \rightarrow H^1(F, \overline{R}^{(m)})$.

To end this subsection, we study two “general image” conditions for integral Galois modules.

Lemma 2.3.3. Let F'/F_+ be a totally real finite Galois extension contained in $\mathbb{R}$ and a polynomial $\mathcal{P}(T) \in \mathbb{Z}[T]$. For each $\alpha \in \{0, 1\}$, we take an object $R_\alpha \in \text{Mod}(F, \mathcal{O}_\lambda)_{\text{fr}}$ with the associated homomorphism $\rho_\alpha : \text{Gal}_F \rightarrow \text{GL}(R_\alpha)$, together with a $(1 - \alpha)$ -polarization $\Xi : R_\alpha^\mathbf{c} \xrightarrow{\sim} R_\alpha^\vee(1 - \alpha)$. We assume that $\text{rank } R_0 = n_0 = 2r_0$ is even and $\text{rank } R_1 = n_1 = 2r_1 + 1$ is odd. Set $R = R_0 \otimes R_1$ and $\Xi : R \xrightarrow{\sim} R^\vee(1)$. For every positive integer $m \in \mathbb{Z}_+$, consider the following statement $(\text{GI}_{R_0, R_1, F', \mathcal{P}}^m)$: The image of the restriction of the homomorphism

$$\left(\overline{\rho}_{R_0, +}^{(m)}, \overline{\rho}_{R_1, +}^{(m)}, \overline{\varepsilon}_\ell^{(m)} \right) : \text{Gal}_F \rightarrow \mathcal{G}_{n_0}(\mathcal{O}_\lambda/\lambda^m) \times \mathcal{G}_{n_1}(\mathcal{O}_\lambda/\lambda^m) \times (\mathcal{O}_\lambda/\lambda^m)^\times$$

(see [LTX⁺22, Notation 2.6.1]) to $\text{Gal}_{F'}$ contains the element $(\gamma_0, \gamma_1, \xi)$ satisfying

(a) $\mathcal{P}(\xi)$ is invertible in $\mathcal{O}_\lambda/\lambda^m$;

(b) for each $\alpha \in \{0, 1\}$, γ_α belongs to $\text{GL}_{n_\alpha}(\mathcal{O}_\lambda/\lambda^m) \times (\mathcal{O}_\lambda/\lambda^m)^\times \times \{\mathbf{c}\}$ with order coprime to ℓ ;

(c) the kernels of $(h_{\gamma_0} - 1)^{n_0}, (h_{\gamma_1} - 1)^{n_1}$ and $(h_{\gamma_0} \otimes h_{\gamma_1} - 1)$ (see [LTX+22, Notation 2.6.2]) are all free over $\mathcal{O}_\lambda/\lambda^m$ of rank 1;

(d) for each $\alpha \in \{0, 1\}$, h_{γ_α} does not have an eigenvalue that is equal to $-\xi$ in κ_λ .

Then $(\mathrm{GI}_{R_0, R_1, F', \mathcal{P}}^1)$ implies $(\mathrm{GI}_{R_0, R_1, F', \mathcal{P}}^m)$ for every $m \in \mathbb{Z}_+$.

Proof. This is [LTX+22, Lemma 2.7.1]. $\square$

Lemma 2.3.4. Let F'/F_+ be a totally real finite Galois extension contained in $\mathbb{R}$ and a polynomial $\mathcal{P}(T) \in \mathbb{Z}[T]$. We take object $R \in \mathrm{Mod}(F, \mathcal{O}_\lambda)_{\mathrm{fr}}$ with the associated homomorphism $\rho : \mathrm{Gal}_F \rightarrow \mathrm{GL}(R)$, together with a 1-polarization $\Xi : R^\vee \xrightarrow{\sim} R^\vee(1)$. We assume that $\mathrm{rank} R = 2r$. For every positive integer $m \in \mathbb{Z}_+$, consider the following statement

$(\mathrm{GI}_{R, F', \mathcal{P}}^m)$: The image of the restriction of the homomorphism

$$\left(\bar{\rho}_{R, +}^{(m)}, \bar{\varepsilon}_\ell^{(m)} \right) : \mathrm{Gal}_F \rightarrow \mathcal{G}_{2r}(\mathcal{O}_\lambda/\lambda^m) \times (\mathcal{O}_\lambda/\lambda^m)^\times$$

(see [LTX+22, Notation 2.6.1]) to $\mathrm{Gal}_{F'}$ contains the element (γ, ξ) satisfying

(a) $\mathcal{P}(\xi)$ is invertible in $\mathcal{O}_\lambda/\lambda^m$;

(b) γ belongs to $\mathrm{GL}_{2r}(\mathcal{O}_\lambda/\lambda^m) \times (\mathcal{O}_\lambda/\lambda^m)^\times \times \{\mathbf{c}\}$ with order coprime to ℓ ;

(c) the kernels of $(h_\gamma - 1)^{2r}$ (see [LTX+22, Notation 2.6.2]) is all free over $\mathcal{O}_\lambda/\lambda^m$ of rank 1;

(d) h_γ does not have an eigenvalue that is equal to $-\xi$ in κ_λ .

Then $(\mathrm{GI}_{R, F', \mathcal{P}}^1)$ implies $(\mathrm{GI}_{R, F', \mathcal{P}}^m)$ for every $m \in \mathbb{Z}_+$.

Proof. The argument of [LTX+22, Lemma 2.7.1] goes through. $\square$

3. THE CONJUGATE SELF-DUAL RANKIN–SELBERG CASE

In this section, we adapt the argument of [LTX+22] to prove Theorem D. While our setup is not identical to that of [LTX+22], we align our notation with theirs whenever possible and record any deviations as they arise. Fix a positive integer $N \geq 2$ and set $r = \lfloor \frac{N}{2} \rfloor$. We work in the following setting.

Setup 3.0.1.

- Let $F_+ \subset \mathbb{R}$ be a totally real number field and let $F \subset \mathbb{C}$ be a quadratic CM extension of F_+ .
- Denote by Σ^∞ (resp. Σ_+^∞) the set of Archimedean places of F (resp. F_+), with τ_∞ (resp. τ_+^∞) the default one induced by the inclusion $F \subset \mathbb{C}$ (resp. $F_+ \subset \mathbb{R}$).
- Let Σ_+^{bad} denote the (finite) set of finite places of F_+ whose underlying rational prime ramifies in F .
- For any place v of F_+ , we set $\mathcal{O}_{F_v} := \mathcal{O}_v \otimes_{\mathcal{O}_{F_+}} \mathcal{O}_F$ and $F_v := F \otimes_{F_+} F_v$.
- For every place w of F with underlying place v of F_+ , we identify Gal_{F_w} with $\mathrm{Gal}_{F_{+,v}} \cap \mathrm{Gal}_F$ (resp. $\mathbf{c}(\mathrm{Gal}_{F_{+,v}} \cap \mathrm{Gal}_F)\mathbf{c}$), if the embedding $\iota_v : \overline{F} \rightarrow \overline{F_{+,v}}$ induces (resp. does not induce) the place w .

3.1. Unitary Satake parameters and unitary Hecke algebras. We recall the notation of the coefficient field for an automorphic representation of $\mathrm{GL}_N(\mathbf{A}_F)$. Let Π be an irreducible relevant automorphic representation of $\mathrm{GL}_N(\mathbf{A}_F)$ that is cuspidal (resp. almost cuspidal) when N is even (resp. N is odd).

Definition 3.1.1. We denote by Σ_+^Π the smallest finite set of (finite) places of F_+ containing Σ_+^{bad} so that Π_w is unramified for every finite place w of F not lying above Σ_+^Π .

Definition 3.1.2. For each ring L , we define an *abstract Satake parameter* in L of rank N to be a multi-set α consisting of N elements in L . For two Satake parameters α, α' in L of dimension n and n' , respectively, we can form their tensor product $\alpha \otimes \alpha'$ in the natural way, which is an abstract Satake parameter of dimension nn' .

Definition 3.1.3.

- For each finite place w of F not lying above Σ_+^Π , let $\alpha(\Pi_w)$ denote the Satake parameter of Π_w , which is an abstract Satake parameter in $\mathbb{C}$ of dimension N (see Definition 3.1.2), and let $\mathbb{Q}(\Pi_w)$ denote the subfield of $\mathbb{C}$ generated by the coefficients of the polynomial

$$\prod_{\alpha \in \alpha(\Pi_w)} \left(T - \alpha \sqrt{\|w\|}^{N-1} \right) \in \mathbb{C}[T].$$

- We define the *coefficient field* of Π to be the compositum of the fields $\mathbb{Q}(\Pi_w)$ for all finite places w of F not lying above Σ_+^Π , denoted by $\mathbb{Q}(\Pi)$,
- For each finite place v of F_+ not in Σ_+^Π and inert in F , the abstract Satake parameter $\alpha(\Pi_v)$ at v of rank N is defined in [LTX⁺22, Notation 3.14], which is an abstract Satake parameter in $\mathbb{C}$ of dimension N .

Definition 3.1.4. Let v be a finite place of F_+ inert in F , L be a ring, and $P \in L[T]$ be a monic polynomial.

- When N is odd, we say P is *Tate generic at v* if $P'(1)$ is invertible in L .
- When N is odd, we say P is *intertwining generic at v* if $P(-\|v\|)$ is invertible in L .
- When N is even, we say P is *level-raising special at v* if $P(\|v\|) = 0$ and $P'(\|v\|)$ is invertible in L .
- When N is even, we say P is *intertwining generic at v* if $P(-1)$ is invertible in L .

Lemma 3.1.5. *The coefficient field $\mathbb{Q}(\Pi)$ is a number field.*

Proof. We take a standard pair $(\mathbf{V}, \pi)$ (in the sense of Definition 3.2.1) such that $\text{BC}(\pi)$ is isomorphic to Π and π_v is unramified for each finite place v of F_+ not in Σ_+^Π . Such a standard pair always exists by Arthur's multiplicity formula [KMSW14, Theorem 1.7.1]. For each finite place v of F_+ , we denote by $\mathbb{Q}(\pi_v)$ the fixed field of the group

$$\{\tau \in \text{Aut}(\mathbb{C}) : \pi_v \otimes_{\mathbb{C}, \tau} \mathbb{C} \cong \pi_v\}.$$

If v is not contained in Σ_+^Π , then $\mathbb{Q}(\text{BC}(\pi_v))$ equals $\mathbb{Q}(\pi_v)$ by [ST14, Lemma 2.25 and Lemma 4.5]. Moreover, the composite field of $\mathbb{Q}(\pi_v)$ for all finite places v not in Σ_+^Π is a number field by [ST14, Proposition 2.15]. Thus the assertion follows. $\square$

Definition 3.1.6. We say a number field $E \subset \mathbb{C}$ is a *strong coefficient field* of Π if E contains $\mathbb{Q}(\Pi)$, and for every finite place λ of E with underlying prime ℓ , there exists a continuous homomorphism

$$\rho_{\Pi, \lambda} : \text{Gal}_F \rightarrow \text{GL}_N(E_\lambda)$$

necessarily unique up to conjugation, such that for every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ inducing the place λ , $\rho_{\Pi, \lambda} \otimes_{E_\lambda} \overline{\mathbb{Q}_\ell}$ and ρ_{Π, ι_ℓ} (see Proposition 2.1.1) are conjugate.

Remark 3.1.7. By the argument of [CH13, Proposition 3.2.5], a strong coefficient field of Π exists. Moreover, if N is odd and Π is almost cuspidal of the form $\Pi = \Pi^\flat \boxplus \mathbf{1}$ where $\mathbf{1}$ is the trivial character of $\text{GL}_1(\mathbf{A}_F)$, then for every strong coefficient E with a finite place λ , the homomorphism $\rho_{\Pi, \lambda}$ is of the form

$$\rho_{\Pi, \lambda} = \rho_{\Pi^\flat, \lambda} \oplus \varepsilon_\ell^{(1-N)/2}.$$

Definition 3.1.8. For any $\mathcal{O}_{F_+}$ -ring R , a *Hermitian space* over $\mathcal{O}_F \otimes_{\mathcal{O}_{F_+}} R$ of dimension N is a projective $\mathcal{O}_F \otimes_{\mathcal{O}_{F_+}} R$ -module V of rank N together with a perfect pairing

$$(-, -)_V : V \times V \rightarrow \mathcal{O}_F \otimes_{\mathcal{O}_{F_+}} R$$

that is $\mathcal{O}_F \otimes_{\mathcal{O}_{F_+}} R$ -linear in the first variable and $(\mathcal{O}_F \otimes_{\mathcal{O}_{F_+}} R, \mathbf{c} \otimes \text{id})$ -linear in the second variable, and satisfies $(x, y)_V = (y, x)_V^\mathbf{c}$ for any $x, y \in V$. We write $\text{U}(V)$ for the group of R -linear isometries of V , which is a reductive group scheme over R .

We denote by $V_\sharp := V \oplus Re$ the orthogonal direct sum Hermitian space where we set $\|e\| = 1$. If $f : V \rightarrow V'$ is an isometry of Hermitian spaces over R , we write $f_\sharp : V_\sharp \rightarrow V'_\sharp$ for the induced isometry of Hermitian spaces over $\mathcal{O}_F \otimes_{\mathcal{O}_{F_+}} R$.

Definition 3.1.9.

- (1) For a finite place v of F_+ not in Σ_+ , let $\Lambda_{N, v}$ denote the unique up to isomorphism Hermitian space over $\mathcal{O}_{F_v}$ of dimension N , and $\text{U}_{N, v}$ its unitary group over $\mathcal{O}_{F_+, v}$. We define spherical Hecke algebra

$$\mathbb{T}_{N, v} := \mathbb{Z}[\text{U}_{N, v}(\mathcal{O}_v) \backslash \text{U}_{N, v}(F_{+, v}) / \text{U}_{N, v}(\mathcal{O}_v)].$$

- (2) For a finite set Σ_+ of finite places of F_+ containing Σ_+^{bad} , we define the abstract unitary Hecke algebra away from Σ_+ to be the restricted tensor product ring

$$\mathbb{T}_N^{\Sigma_+} := \bigotimes_v \mathbb{T}_{N, v}$$

over all finite places of F_+ not in Σ_+ , with respect to the unit elements.

- (3) The Hecke character $\phi_\Pi : \mathbb{T}_N^{\Sigma_\Pi^+} \rightarrow \mathbb{C}$ attached to Π is defined in [LTX⁺22, Construction 3.1.10]. By [BG14, Lemma 2.2.3], ϕ_Π takes value in $\mathbb{Q}(\Pi)$. Furthermore, ϕ_Π takes values in $\mathcal{O}_{\mathbb{Q}(\Pi)}$. In fact, if we take a standard pair $(\mathbf{V}, \pi)$ (in the sense of Definition 3.2.1) such that $\text{BC}(\pi)$ is isomorphic to Π and π_v is unramified for each finite place v of F_+ not in Σ_Π^+ . Such a standard pair always exists by Arthur's multiplicity formula [KMSW14, Theorem 1.7.1]. Then ϕ_Π is identical to the spherical Hecke character of π , which is easily seen to be valued in algebraic integers.

3.2. Unitary Shimura varieties. Let $\mathbf{V}$ be a Hermitian space over F of dimension N .

Definition 3.2.1.

- (1) Recall from [LTX⁺22, Definition 3.1.11] that, for any finite set Σ_+ of finite places of F_+ we have
- (a) a category $\mathfrak{K}(\mathbf{V})^{\Sigma_+}$ whose objects are neat compact open subgroups of $\text{U}(\mathbf{V})(\mathbf{A}_{F_+}^{\infty, \Sigma})$ and whose morphisms are double cosets. There is also a subcategory $\mathfrak{K}(\mathbf{V})^{\Sigma_+}$ of $\mathfrak{K}(\mathbf{V})^{\Sigma_+}$ consisting of the same objects but allowing only identity double cosets; and
 - (b) a category $\mathfrak{K}(\mathbf{V})_{\text{sp}}^{\Sigma_+}$ consisting of pairs $(\mathcal{K}_b, \mathcal{K}_\#)$, where $\mathcal{K}_b$ (resp. $\mathcal{K}_\#$) is an object of $\mathfrak{K}(\mathbf{V})^{\Sigma_+}$ (resp. $\mathfrak{K}(\mathbf{V}_\#)^{\Sigma_+}$) such that $\mathcal{K}_b$ is contained in $\mathcal{K}_\#$. There are the obvious functors

$$(-)_b : \mathfrak{K}(\mathbf{V})_{\text{sp}}^{\Sigma_+} \rightarrow \mathfrak{K}(\mathbf{V})^{\Sigma_+}, \quad (-)_\# : \mathfrak{K}(\mathbf{V})_{\text{sp}}^{\Sigma_+} \rightarrow \mathfrak{K}(\mathbf{V}_\#)^{\Sigma_+}.$$

When Σ_+ is the empty set, we suppress it from all the notations above.

- (2) We say $\mathbf{V}$ is standard definite if it has signature $(N, 0)$ at each real place of F_+ . We say $\mathbf{V}$ is standard indefinite if it has signature $(N - 1, 1)$ at τ_∞ and $(N, 0)$ at other real places of F_+ .
- (3) For a discrete automorphic representation π of $\text{U}(\mathbf{V})(\mathbf{A}_{F_+})$, we say $(\mathbf{V}, \pi)$ is a standard pair if one of the following holds:
- (a) $\mathbf{V}$ is standard definite, and π^∞ appears in

$$\lim_{\mathcal{K} \in \mathfrak{K}'(\mathbf{V})} \mathbb{C}[\text{Sh}(\mathbf{V}, \mathcal{K})].$$

- (b) $\mathbf{V}$ is standard indefinite, and π^∞ appears in

$$\lim_{\mathcal{K} \in \mathfrak{K}'(\mathbf{V})} \iota_\ell^{-1} \text{H}_{\text{ét}}^i(\text{Sh}(\mathbf{V}, \mathcal{K})_{\overline{F}}, \overline{\mathbb{Q}_\ell}),$$

for some rational prime ℓ with isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ and some $i \in \mathbb{N}$.

Proposition 3.2.2. *Let π be a discrete automorphic representation of $\text{U}(\mathbf{V})(\mathbf{A}_{F_+})$ such that $(\mathbf{V}, \pi)$ is a standard pair. For every rational prime ℓ and every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, there exists a semisimple continuous homomorphism*

$$\rho_{\text{BC}(\pi), \iota_\ell} : \text{Gal}_F \rightarrow \text{GL}_N(\overline{\mathbb{Q}_\ell}),$$

unique up to conjugation, satisfying that

$$(3.1) \quad \text{WD}_\ell(\rho_{\text{BC}(\pi), \iota_\ell}|_{\text{Gal}_{F_w}})^{\text{F-ss}} \cong \iota_\ell \text{rec}_N \left(\text{BC}(\pi)_w \otimes |\det|^{\frac{1-N}{2}} \right),$$

for every finite place w of F , where rec_N is the local Langlands correspondence for $\text{GL}_N(F_w)$. Moreover, ρ_{Π, ι_ℓ}^c and $\rho_{\Pi, \iota_\ell}^\vee(1 - N)$ are conjugate.

Proof. This follows from Arthur's multiplicity formula [CZ24, Theorem 2.6] and standard results, see for example [CH13, Theorem 3.2.3], [Car12, Theorem 1.1] and [Car12, Theorem 1.1]. $\square$

When $\mathbf{V}$ is standard definite (resp. standard indefinite), there are functors $\text{Sh}(\mathbf{V}, -) : \mathfrak{K}(\mathbf{V}) \rightarrow \text{Set}$ (resp. $\text{Sh}(\mathbf{V}, -) : \mathfrak{K}(\mathbf{V}) \rightarrow \text{Sch}/F$) of Shimura sets (Shimura varieties) attached to $\text{Res}_{F_+/\mathbb{Q}} \text{U}(\mathbf{V})$, as defined in [LTX⁺22, §3.2].

Hypothesis 3.2.3. Suppose $\mathbf{V}$ is a standard indefinite Hermitian space over F of dimension N , and π is a discrete automorphic representation of $\text{U}(\mathbf{V})(\mathbf{A}_{F_+})$ such that the functorial lift $\text{BC}(\pi)$ is a relevant automorphic representation of $\text{GL}_N(\mathbf{A}_F)$ (see Definition 4.4.2). For every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, we consider the $\overline{\mathbb{Q}_\ell}[\text{Gal}_F]$ -module

$$W^{N-1}(\pi^\infty) := \text{Hom}_{\overline{\mathbb{Q}_\ell}[\text{U}(\mathbf{V})(\mathbf{A}_{F_+}^\infty)]} \left(\iota_\ell \pi^\infty, \varinjlim_{\mathfrak{K}'(\mathbf{V})} \text{H}_{\text{ét}}^{N-1}(\text{Sh}(\mathbf{V}, \mathcal{K})_{\overline{F}}, \overline{\mathbb{Q}_\ell}) \right).$$

- (1) If $\rho_{\text{BC}(\pi), \iota_\ell}$ is irreducible, then $W^{N-1}(\pi^\infty)$ is isomorphic to $\rho_{\text{BC}(\pi), \iota_\ell}^\epsilon$.
- (2) If N is odd, $\text{BC}(\pi) = \Pi^\flat \boxplus \chi$ is almost cuspidal, and $\rho_{\Pi^\flat, \iota_\ell}$ is irreducible, then $W^{N-1}(\pi^\infty)$ is isomorphic to either $\rho_{\Pi^\flat, \iota_\ell}^\epsilon$ or $\rho_{\chi, \iota_\ell}^\epsilon$. Moreover, if there is a finite place w of F over a place of F_+ inert in F such that $\Pi_w^\flat$ is square-integrable, then there exists a unique irreducible admissible representation π_1^∞ of $\text{U}(\mathbf{V})(\mathbf{A}_F^\infty)$ such that π_1^∞ is isomorphic to π^∞ away from w , and $W^{N-1}(\pi^\infty) \oplus W^{N-1}(\pi_1^\infty)$ is conjugate to $\rho_{\Pi, \iota_\ell}^\epsilon$ as Gal_F -representations.

Proposition 3.2.4. Hypothesis 3.2.3 holds if $N \leq 3$ or $F_+ \neq \mathbb{Q}$.

Proof. The case for $N = 2$ is established by Liu [Liu21, Theorem D.6]. The case for $N = 3$ and $F_+ = \mathbb{Q}$ follows from the main result of [Rog92]. The case for $N \geq 3$ when $F_+ \neq \mathbb{Q}$ will be established in a sequel to [KSZ21], assuming the full endoscopic classification for unitary groups. Note that the full endoscopic classification for such unitary groups is established by Chen-Zou [CZ24, Corollary 7.6]. $\square$

We recall the following definition of cohomological Hecke characters from [LTX⁺22].

Definition 3.2.5. Let $N \in \mathbb{Z}_+$ be a positive integer, and Σ_+ a finite set of finite places of F_+ containing Σ_+^{bad} . Consider a homomorphism $\phi : \mathbb{T}_{N^+}^{\Sigma_+} \rightarrow \kappa$ with κ a field. We say ϕ is *cohomologically generic* if

$$H_{\text{ét}}^i(\text{Sh}(\mathbf{V}, \mathcal{K})_{\overline{F}}, \kappa)_{\mathbb{T}_{N^+}^{\Sigma_+} \cap \ker \phi} = 0$$

holds for any tuple $(\Sigma'_+, i, \mathbf{V}, \mathcal{K})$ in which

- Σ'_+ is a finite set of finite places of F_+ containing Σ_+ ,
- i is a nonnegative integer distinct from $N - 1$,
- $\mathbf{V}$ is a standard indefinite Hermitian space over F of dimension N , and
- $\mathcal{K}$ is an object of $\mathfrak{K}(\mathbf{V})$ of the form $\mathcal{K} = \mathcal{K}_{\Sigma'_+} \times \prod_{v \in \Sigma_{F_+}^{\text{fin}} \setminus \Sigma'_+} \text{U}(\Lambda)(\mathcal{O}_v)$ for some self-dual

$$\prod_{v \in \Sigma_{F_+}^{\text{fin}} \setminus \Sigma'_+} \mathcal{O}_{F_v}\text{-lattice } \Lambda \text{ in } \mathbf{V} \otimes_{F_+} \mathbf{A}_{F_+}^{\Sigma_{F_+}^{\text{fin}} \setminus \Sigma'_+}.$$

3.3. Generalized CM type and reflexive closure. We denote by $\mathbb{N}[\Sigma_F^\infty]$ the commutative monoid freely generated by the set Σ_F^∞ , which admits an action of $\text{Aut}(\mathbb{C})$ via the set Σ_F^∞ .

Definition 3.3.1. A *generalized CM type* of rank N is an element

$$\Psi = \sum_{\tau \in \Sigma_\infty} r_\tau \tau \in \mathbb{N}[\Sigma_F^\infty]$$

satisfying $r_\tau + r_{\tau^c} = N$ for every $\tau \in \Sigma_F^\infty$. For such Ψ , we define its reflex field $F_\Psi \subset \mathbb{C}$ to be the fixed subfield of the stabilizer of Ψ in $\text{Aut}(\mathbb{C})$. A *CM type* is simply a generalized CM type of rank 1.

Definition 3.3.2. We define the *reflexive closure* of F , denoted by F_{rflx} , to be the subfield of $\mathbb{C}$ generated by F and the intersections of F_Φ for all CM types Φ of F . Set $F_{\text{rflx},+} := (F_{\text{rflx}})^{c=1}$.

Definition 3.3.3. We say a finite place $\mathfrak{p}$ of F_+ is *good inert* if it is inert in F and splits completely in $F_{\text{rflx},+}$. By abuse of notation, we also denote by $\mathfrak{p}$ the induced finite place of F . We say a good inert place $\mathfrak{p}$ is *very good inert* if the following are satisfied:

- (1) the underlying rational prime p of $\mathfrak{p}$ is odd and unramified in F ;
- (2) $\mathfrak{p}$ is of degree one over $\mathbb{Q}$, that is, $F_{+, \mathfrak{p}} = \mathbb{Q}_p$.

Remark 3.3.4. A finite place $\mathfrak{p}$ of F_+ is very good inert in our sense if it is very special inert in the sense of [LTX⁺22, Definition 3.3.4].

3.4. Preparation for Tate classes and arithmetic level-raising. We now work in the following setting.

Setup 3.4.1.

- Let Π be a relevant representation of $\text{GL}_N(\mathbf{A}_F)$ that is cuspidal (resp. almost cuspidal of the form $\Pi = \Pi^\flat \boxplus \mathbf{1}$) if N is even (resp. odd). Here $\mathbf{1}$ is the trivial character of $\text{GL}_1(\mathbf{A}_F)$.
- Let $E \subset \mathbb{C}$ be a strong coefficient field of Π (see Definition 3.1.6).
- Let Σ_+^{min} be a finite set of finite places of F_+ that contains Σ_+^Π (see Definition 3.1.1).

- Let λ be a finite place of E whose underlying prime ℓ satisfies $\Sigma_+^{\min} \cap \Sigma_{F_+}(\ell) = \emptyset$. We fix an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ that induces the place λ .
- Let Σ_+^{lr} be a finite set of finite places of F_+ that are inert in F , which is strongly disjoint from $\Sigma_+^{\min}$ and satisfies $\ell \nmid \|v\| (\|v\|^2 - 1)$ for any $v \in \Sigma_+^{\text{lr}}$.
- Let Σ_+ be a finite set of finite places of F_+ containing $\Sigma_+^{\min}$ and Σ_+^{lr} .
- Let $\phi_\Pi : \mathbb{T}_N^{\Sigma_+} \rightarrow \mathcal{O}_E$ be the Hecke character attached to Π (see Definition 3.1.9).
- Let $\rho_{\Pi, \lambda} : \text{Gal}_F \rightarrow \text{GL}_N(E_\lambda)$ be the continuous homomorphism attached to Π (see Definition 3.1.6). In particular, $\rho_{\Pi, \lambda}^\vee$ and $\rho_{\Pi, \lambda}^\vee(1 - N)$ are conjugate.
- Let $\mathcal{V}_N^\circ = (\mathbf{V}_N^\circ, \Lambda_N^\circ, \mathcal{K}_N^\circ)$ be a triple, where³
 - (1) $\mathbf{V}_N^\circ$ is a standard definite Hermitian space over F of dimension N (see Definition 3.2.1) such that $(\mathbf{V}_N^\circ)_v$ is not split for $v \in \Sigma_+^{\text{lr}}$ when N is even;
 - (2) Λ_N° is a self-dual $\prod_{v \in \Sigma_{F_+}^{\text{fin}} \setminus \Sigma_+^{\min}} \mathcal{O}_{F_v}$ -lattice in $\mathbf{V}_N^\circ \otimes_{F_+} \mathbf{A}_{F_+}^{\infty, \Sigma^{\min}}$;
 - (3) $\mathcal{K}_N^\circ$ is an object in $\mathfrak{K}(\mathbf{V}_N^\circ)$ of the form

$$\mathcal{K}_N^\circ = \prod_{v \in \Sigma_+} (\mathcal{K}_N^\circ)_v \times \prod_{v \in \Sigma_{F_+}^{\text{fin}} \setminus \Sigma_+} \text{U}(\Lambda_N^\circ)(\mathcal{O}_v),$$

satisfying that when N is even, $(\mathcal{K}_N^\circ)_v$ is a hyperspecial maximal subgroup of $\text{U}(\mathbf{V}_N^\circ)(F_v)$ for $v \in \Sigma_+ \setminus (\Sigma_+^{\text{lr}} \cup \Sigma_+^{\min})$, and is a special maximal subgroup of $\text{U}(\mathbf{V}_N^\circ)(F_v)$ for $v \in \Sigma_+^{\text{lr}}$ such that

$$\frac{\mathcal{O}_\lambda[\text{Sh}(\mathbf{V}_N^\circ, \mathcal{K}_N^\circ)]}{\mathbb{T}_N^{\Sigma_+} \cap \ker \phi_{\Pi_N}}$$

is nontrivial when N is even.

- Let $m \in \mathbb{Z}_+$ be a positive integer.
- Let $\mathfrak{p}$ be a very good inert place of F_+ with the underlying rational prime p (see Definition 3.3.3), satisfying⁴
 - (P1) $\mathfrak{p}$ is strongly disjoint from Σ_+ ;
 - (P2) ℓ does not divide $p(p^2 - 1)$;
 - (P3) There exists a CM type Φ containing τ_∞ with $\mathbb{Q}_{p^2}^\Phi = \mathbb{Q}_{p^2}$ (we refer to [LTX⁺22, §3.3] for the definitions).
 - (P4) If N is even, then $P_{\alpha(\Pi_{\mathfrak{p}})}(\text{mod } \lambda^m)$ is level-raising special at $\mathfrak{p}$; if N is odd, then $P_{\alpha(\Pi_{\mathfrak{p}})}(\text{mod } \lambda)$ is Tate generic at $\mathfrak{p}$ (see Definition 3.1.4);
 - (P5) $P_{\alpha(\Pi_{\mathfrak{p}})}(\text{mod } \lambda)$ is intertwining generic at $\mathfrak{p}$.

In particular, we can and will apply the construction and notations in [LTX⁺22, §5.1] to the datum $(\mathbf{V}_N^\circ, \{\Lambda_{N, \mathfrak{q}}^\circ\}_{|\mathfrak{q}|p})$; cf. the beginning of [LTX⁺22, §5.2]. Denote by

$$\mathfrak{m} := \mathbb{T}_N^{\Sigma_+ \cup \Sigma_{F_+}(p)} \cap \ker \left(\mathbb{T}_N^{\Sigma_+} \xrightarrow{\phi_\Pi} \mathcal{O}_E \rightarrow \mathcal{O}_E/\lambda \right)$$

and

$$\mathfrak{n} := \mathbb{T}_N^{\Sigma_+ \cup \Sigma_{F_+}(p)} \cap \ker \left(\mathbb{T}_N^{\Sigma_+} \xrightarrow{\phi_\Pi} \mathcal{O}_E \rightarrow \mathcal{O}_E/\lambda^m \right)$$

the two ideals of $\mathbb{T}_N^{\Sigma_+ \cup \Sigma_{F_+}(p)}$.

- Let $\mathcal{T} = (\Phi, \mathbf{W}_0, \mathcal{K}_0^p, \iota_p, \varpi)$ be a quintuple of data as in [LTX⁺22, §5.1] with $\mathbb{Q}_p^\Phi = \mathbb{Q}_{p^2}$, which is possible because $\mathfrak{p}$ is very good inert.
- Let $\Lambda_{N, \mathfrak{p}}^\bullet$ be a lattice in $\mathbf{V}_N^\circ \otimes_F F_{\mathfrak{p}}$ satisfying
 - $\Lambda_{N, \mathfrak{p}}^\bullet \subset \Lambda_{N, \mathfrak{p}}^\bullet \subset p^{-1} \Lambda_{N, \mathfrak{p}}^\bullet$, and

³Compared with [LTX⁺22, §6.1], we omit the assumption that $(\mathcal{K}_N^\circ)_v$ is transferable when N is even, which is possible by [LTX⁺25, Remark 8.2]

⁴Compared with [LTX⁺22, §6.1], we omit the assumption (PI6), because it will be redundant for applications in view of [LTX24, Lemma 4.2.4(2)].

- $p\Lambda_{N,\mathfrak{p}}^\bullet \subset \Lambda_{N,\mathfrak{p}}^{\bullet,\vee}$ and $\Lambda_{N,\mathfrak{p}}^{\bullet,\vee}/p\Lambda_{N,\mathfrak{p}}^\bullet$ has length $\frac{1-(-1)^N}{2}$.
- Let $\mathcal{K}_{N,\mathfrak{p}}^\bullet$ denote the stabilizer of $\Lambda_{N,\mathfrak{p}}^\bullet$ in $U(\mathbf{V}'_N \otimes_F F_{\mathfrak{p}})$, and set $\mathcal{K}_{N,p}^\bullet := \mathcal{K}_{N,\mathfrak{p}}^\bullet \times \prod_{\mathfrak{q} \in \Sigma_{F_+}(p) \setminus \{\mathfrak{p}\}} \mathcal{K}_{\mathfrak{q}}^\circ$.
- Let $\mathcal{U}_N = (\mathbf{V}'_N, \{\Lambda'_{N,\mathfrak{q}}\}_{\mathfrak{q}|p}, \mathcal{K}'_{N,p}, j_N)$ be an indefinite uniformization datum for $\mathbf{V}_N^\circ$, which means
 - $\mathbf{V}'_N$ is a standard indefinite Hermitian space over F of dimension N ;
 - for every place $\mathfrak{q}$ of F_+ lying above p other than $\mathfrak{p}$, $\Lambda'_{N,\mathfrak{q}}$ is a self-dual $\mathcal{O}_{F_{\mathfrak{q}}}$ -lattice in $\mathbf{V}' \otimes_F F_{\mathfrak{q}}$;
 - $\Lambda'_{N,\mathfrak{p}}$ is an $\mathcal{O}_{F_{\mathfrak{p}}}$ -lattice in $\mathbf{V}' \otimes_F F_{\mathfrak{p}}$ satisfying $\Lambda'_{N,\mathfrak{p}} \subset (\Lambda'_{N,\mathfrak{p}})^\vee$ and $(\Lambda'_{N,\mathfrak{p}})^\vee/\Lambda'_{N,\mathfrak{p}}$ has length 1;
 - $\mathcal{K}'_{N,p} = \prod_{\mathfrak{q} \in \Sigma_{F_+}(p)} \mathcal{K}'_{N,\mathfrak{q}}$, where $\mathcal{K}'_{N,\mathfrak{q}}$ is the stabilizer of $\Lambda'_{N,\mathfrak{q}}$ in $U(\mathbf{V}'_N \otimes_F F_{\mathfrak{q}})$ for each $\mathfrak{q} \in \Sigma_{F_+}(p)$; and
 - $j_N : \mathbf{V}_N^\circ \otimes_{\mathbb{Q}} \mathbf{A}^{\infty,p} \rightarrow \mathbf{V}'_N \otimes_{\mathbb{Q}} \mathbf{A}^{\infty,p}$ is an isometry.
- Set $\mathcal{K}_N^{p,\circ} := (\mathcal{K}_N^\circ)^p$, and $\mathcal{K}_N^\bullet := \mathcal{K}_N^{p,\circ} \times \mathcal{K}_{N,p}^\bullet$.
- Set $X_N^? := X_{\mathfrak{p}}^?(\mathbf{V}_N^\circ, \mathcal{K}_N^{p,\circ})$ for meaningful pairs $(X, ?) \in \{\mathbf{M}, \mathbf{M}, \mathbf{B}, \mathbf{S}\} \times \{\eta, \circ, \bullet, \dagger\}$, and let $(E_s^{p,q}, d_s^{p,q})$ denote the weight spectral sequence abutting to the cohomology $H_{\mathfrak{T}}^\bullet(\overline{\mathbf{M}}_N, R\Psi\mathcal{O}_\lambda(r))$ from [LTX⁺22, §5.9].

Assumption 3.4.2. The composite homomorphism $\mathbb{T}_N^{\Sigma_+} \xrightarrow{\phi_\Pi} \mathcal{O}_E \rightarrow \kappa_\lambda$ is cohomologically generic (see Definition 3.2.5).

Assumption 3.4.3. The Galois representation $\rho_{\Pi,\lambda}$ (resp. $\rho_{\Pi^p,\lambda}$) is residually absolutely irreducible when N is even (resp. N is odd).

Under Assumption 3.4.3, we get a residual representation $\overline{\rho}_{\Pi,\lambda}$, which is unique up to conjugation and $(1-N)$ -polarizable in the sense of [LTX⁺22, Definition 2.5.3]. Then we obtain a continuous homomorphism

$$(3.2) \quad \overline{\rho}_{\Pi,\lambda,+} : \text{Gal}_{F_+} \rightarrow \mathcal{G}_N(\kappa_\lambda)$$

from [LTX⁺22, Construction 2.5.4].

Definition 3.4.4. We say a standard pair $(\mathbf{V}, \pi)$ (see Definition 3.2.1) with $\dim_F \mathbf{V} = N$ is Π -congruent (outside $\Sigma_+ \cup \Sigma_{F_+}(p)$) if for any finite place v of F_+ not in $\Sigma_+ \cup \Sigma_{F_+}(\{p, \ell\})$, π_v is unramified, and the two homomorphisms $\iota_\ell \phi_{\alpha(\text{BC}(\pi)_v)}$ and $\iota_\ell \phi_{\alpha(\Pi_v)}$ from $\mathbb{T}_{N,v}$ to $\overline{\mathbb{Q}}_\ell$, taking values in $\overline{\mathbb{Z}}_\ell$, coincide in $\overline{\mathbb{F}}_\ell$.

Lemma 3.4.5. Assume Assumption 3.4.3. Then the natural maps

$$\begin{aligned} H_{\text{ét},c}^i \left(\text{Sh}(\mathbf{V}'_N, j_N \mathcal{K}_N^{p,\circ} \mathcal{K}'_{p,N})_{\overline{F}}, \mathcal{O}_\lambda \right)_{\mathfrak{m}} &\rightarrow H_{\text{ét}}^i \left(\text{Sh}(\mathbf{V}'_N, j_N \mathcal{K}_N^{p,\circ} \mathcal{K}'_{p,N})_{\overline{F}}, \mathcal{O}_\lambda \right)_{\mathfrak{m}} \\ H_{\mathfrak{T},c}^i \left(\overline{\mathbf{M}}_N^\bullet, \mathcal{O}_\lambda \right)_{\mathfrak{m}} &\rightarrow H_{\mathfrak{T}}^i \left(\overline{\mathbf{M}}_N^\bullet, \mathcal{O}_\lambda \right)_{\mathfrak{m}} \end{aligned}$$

are both isomorphisms for every $i \in \mathbb{N}$.

Proof. We follow the proof of [LTX⁺22, Lemma 6.1.11]. We abbreviate $\text{Sh} := \text{Sh}(\mathbf{V}'_N, j_N \mathcal{K}_N^{p,\circ} \mathcal{K}'_{p,N})$. By [LTX⁺22, Lemma 5.2.7] and the description of the weight spectral sequences $(E_s^{p,q}, d_s^{p,q})$ in [LTX⁺22, Lemma 5.9.2] (for N odd) and [LTX⁺22, Lemma 5.9.3] for N even, it suffices to show that the first map is an isomorphism for every $i \in \mathbb{N}$. This is trivial if $F_+ \neq \mathbb{Q}$, because in that case Sh is proper.

If $F_+ = \mathbb{Q}$, then the Witt index of $\mathbf{V}'_N$ is 1. In that case, the Shimura variety Sh has a unique toroidal compactification [AMRT75], which we denote by $\widetilde{\text{Sh}}$. Since the choice of the relevant combinatorial data is unique, $\widetilde{\text{Sh}}$ is smooth over F . As $j_N \mathcal{K}_N^{p,\circ} \mathcal{K}'_{p,N}$ is neat, the boundary $Z := \widetilde{\text{Sh}} \setminus \text{Sh}$ is geometrically isomorphic to a disjoint union of abelian varieties of dimension $N-2$. In particular, $H_{\text{ét}}^i(Z_{\overline{F}}, \mathcal{O}_F)$ is a finite free $\mathcal{O}_\lambda$ -module. Let π'^∞ be an irreducible admissible representation of $U(\mathbf{V}'_N)(\mathbf{A}_{F_+}^\infty)$ that appears in $H_{\text{ét}}^i(Z_{\overline{F}}, \mathcal{O}_\lambda) \otimes_{\mathcal{O}_{\lambda,\ell^{-1}}} \mathbb{C}$. Then π'^∞ extends to an automorphic representation π' of $U(\mathbf{V}'_N)(\mathbf{A}_{F_+})$ that is a subquotient of the parabolic induction of a cuspidal automorphic representation $\pi_{\mathbf{L}}$ of $\mathbf{L}(\mathbf{A}_{F_+})$ where $\mathbf{L}$ is the unique proper Levi subgroup of $U(\mathbf{V}'_N)$ up to conjugation.

We write $\mathbf{L} = U(\mathbf{V}_{N-2}) \times \text{Res}_{F/F_+} \text{GL}_1$, where $\mathbf{V}_{N-2}$ is a standard definite Hermitian spaces of dimension $N-2$ contained in $\mathbf{V}_N$ (if $N=2$, $U(\mathbf{V}_{N-2})$ denotes the trivial group). Then we can write $\pi_{\mathbf{L}} = \pi_{N-2} \boxtimes \chi$ where π_{N-2} is a cuspidal automorphic representation of $U(\mathbf{V}_{N-2})$ and χ is an automorphic character of $\text{GL}_1(\mathbf{A}_F)$. In particular, $\text{BC}(\pi')$ is of the form $\text{BC}(\pi') = \text{BC}(\pi_{N-2}) \boxplus \chi \boxplus \chi^{-1}$. Then it is impossible that the (semi-simplified) residual representation of $\rho_{\text{BC}(\pi')}$ is conjugate to that of ρ_Π , as the latter has

an irreducible component of at least dimension $\max(2, N - 1)$ (Note that 2 is even). Thus $H_{\text{ét}}^i(Z_{\overline{F}}, \mathcal{O}_{\lambda})_{\mathfrak{m}}$ vanishes, because for any automorphic representation π such that π^{∞} appearing in $H_{\text{ét}}^i(Z_{\overline{F}}, \mathcal{O}_{\lambda})_{\mathfrak{m}} \otimes_{\mathcal{O}_{\lambda}, \iota_{\ell}^{-1}} \mathbb{C}$, $\rho_{\text{BC}(\pi)}$ should have (semi-simplified) residual representation conjugate to that of ρ_{Π} . This implies that

$$H_{\text{ét},c}^i(\text{Sh}, \mathcal{O}_{\lambda})_{\mathfrak{m}} \xrightarrow{\sim} H_{\text{ét}}^i(\text{Sh}, \mathcal{O}_{\lambda})_{\mathfrak{m}}$$

is an isomorphism for every $i \in \mathbb{N}$. $\square$

Lemma 3.4.6. *Let $(\mathbf{V}, \pi)$ be a Π -congruent standard pair. If Assumption 3.4.3 holds, then $\text{BC}(\pi)$ is a relevant automorphic representation of $\text{GL}_N(\mathbf{A}_F)$ (see Definition 1.1.3).*

Proof. Let $\rho_{\text{BC}(\pi), \iota_{\ell}} : \text{Gal}_F \rightarrow \text{GL}_N(\overline{\mathbb{Q}_{\ell}})$ denote the Galois representation attached to π (see Proposition 3.2.2). Since $(\mathbf{V}, \pi)$ is Π -congruent, by the Chebotarev density theorem, $\rho_{\text{BC}(\pi), \iota_{\ell}}$ admits a lattice whose semisimplified residual representation $\overline{\rho}_{\text{BC}(\pi), \iota_{\ell}}$ is isomorphic to $\overline{\rho}_{\Pi, \lambda} \otimes_{\kappa_{\lambda}} \overline{\mathbb{F}_{\ell}}$, which is irreducible (resp. the sum of an irreducible Galois representation with a character) if N is even (resp. odd). If N is even, then $\rho_{\text{BC}(\pi), \iota_{\ell}}$ is irreducible, so $\text{BC}(\pi)$ must be cuspidal. If N is odd, then $\rho_{\text{BC}(\pi), \iota_{\ell}}$ is either irreducible or a sum of a character and an irreducible Galois representation. In the former case, $\text{BC}(\pi)$ is cuspidal and conjugate self-dual. Assume now that N is odd and $\text{BC}(\pi)$ is not cuspidal. Then $\text{BC}(\pi)$ must be the isobaric sum of a conjugate self-dual cuspidal automorphic representation of $\text{GL}_{N-1}(\mathbf{A}_F)$ and a conjugate self-dual character χ of $\text{GL}_1(\mathbf{A}_F)$.

We now show that $\text{BC}(\pi)$ is relevant. By the above argument, it suffices to show that $\text{BC}(\pi)_w$ is isomorphic to Π_w for every infinite place w of F . Let

$$H_{(2)}^i(\text{Sh}) := \varprojlim_{\mathcal{K} \in \mathcal{R}'(\mathbf{V})} H_{(2)}^i(\text{Sh}(\mathbf{V}, \mathcal{K}), \mathbb{C})$$

be the L^2 -cohomology as defined in [Fal83, §6], It follows from (an analogue of) Lemma 3.4.5 that there are isomorphisms

$$\lim_{\mathcal{K} \in \mathcal{R}'(\mathbf{V})} \iota_{\ell}^{-1} H_{\text{ét},c}^i(\text{Sh}(\mathbf{V}, \mathcal{K})_{\overline{F}}, \overline{\mathbb{Q}_{\ell}})_{\mathfrak{m}} \cong \iota_{\ell} H_{(2)}^i(\text{Sh})_{\mathfrak{m}} \cong \lim_{\mathcal{K} \in \mathcal{R}'(\mathbf{V})} \iota_{\ell}^{-1} H_{\text{ét}}^i(\text{Sh}(\mathbf{V}, \mathcal{K})_{\overline{F}}, \overline{\mathbb{Q}_{\ell}})_{\mathfrak{m}}.$$

In particular, π^{∞} appears in $\iota_{\ell} H_{(2)}^i(\text{Sh})$. By Borel–Casselman’s decomposition of $H_{(2)}^i(\text{Sh})$ [BC83], we see that π_{∞} is cohomological for the trivial representation of $\text{Res}_{F_+/\mathbb{Q}} \text{U}(\mathbf{V})$. In particular, $\text{BC}(\pi)_w$ is isomorphic to Π_w for every infinite place w of F . $\square$

Lemma 3.4.7. *Let N be odd and assume Assumption 3.4.2 and Hypothesis 3.2.3 for N . Then for any object $\mathcal{K}_N^{\text{p}} \in \mathcal{R}(\mathbf{V}'_N)^{\text{p}}$ and hyperspecial maximal subgroup $\mathcal{K}_N^{\text{hs}}$ of $\text{U}(\mathbf{V}'_N)(F_{+, \text{p}})$, there are isomorphisms*

$$H_{\text{ét}}^i(\text{Sh}(\mathbf{V}'_N, \mathcal{K}_N^{\text{p}} \mathcal{K}'_{N, \text{p}})_{\overline{F}}, \mathcal{O}_{\lambda})_{\mathfrak{m}} \cong H_{\text{ét}}^i(\text{Sh}(\mathbf{V}'_N, \mathcal{K}_N^{\text{p}} \mathcal{K}_{N, \text{p}}^{\text{hs}})_{\overline{F}}, \mathcal{O}_{\lambda})_{\mathfrak{m}}$$

Proof. As both $\mathcal{K}'_{N, \text{p}}$ and $\mathcal{K}_{N, \text{p}}^{\text{hs}}$ are special maximal subgroups of $\text{U}(\mathbf{V}'_N)(F_{+, \text{p}})$, the proof of [LTX⁺22, Lemma 8.1.7] goes through, noticing that for every cuspidal automorphic representation π' of $\text{U}(\mathbf{V}')(\mathbf{A}_{F_+})$ appearing in either

$$H_{\text{ét}}^i(\text{Sh}(\mathbf{V}'_N, \mathcal{K}_N^{\text{p}} \mathcal{K}'_{N, \text{p}})_{\overline{F}}, \mathcal{O}_{\lambda})_{\mathfrak{m}} \otimes_{\mathcal{O}_{\lambda}} \overline{\mathbb{Q}_{\ell}}$$

or

$$H_{\text{ét}}^i(\text{Sh}(\mathbf{V}'_N, \mathcal{K}_N^{\text{p}} \mathcal{K}_{N, \text{p}}^{\text{hs}})_{\overline{F}}, \mathcal{O}_{\lambda})_{\mathfrak{m}} \otimes_{\mathcal{O}_{\lambda}} \overline{\mathbb{Q}_{\ell}},$$

the semisimplified residual representations of $\rho_{\text{BC}(\pi'), \iota_{\ell}}$ and $\rho_{\Pi, \iota_{\ell}}$ are conjugate as $\overline{\mathbb{F}_{\ell}}[\text{Gal}_F]$ -modules by the Chebotarev density theorem. $\square$

3.5. Tate classes in the odd rank case. In this subsection, we assume that N is odd, and work in the setting of Setup 3.4.1.

Lemma 3.5.1. *$H_{\mathfrak{I}}^i(\overline{\mathbf{M}}_N^{\dagger}, \mathcal{O}_{\lambda})_{\mathfrak{m}}$ vanishes for every odd integer i .*

Proof. We follow the proof of [LTX⁺22, Lemma 6.2.1]. If $i \neq 2r - 1$, this follows from [LTX⁺22, Lemma 5.6.2(1)]. We now assume $i = 2r - 1$.

Suppose that $\pi^{\infty, \text{p}}$ is an irreducible admissible representation of $\text{U}(\mathbf{V}_N^{\circ})(\mathbf{A}_{F_+}^{\infty, \text{p}})$ that appears in the cohomology $H_{\mathfrak{I}}^{2r-1}(\overline{\mathbf{M}}_N^{\dagger}, \mathcal{O}_{\lambda})_{\mathfrak{m}} \otimes_{\mathcal{O}_{\lambda}, \iota_{\ell}^{-1}} \mathbb{C}$. By [LTX⁺22, Proposition 5.6.4], we may complete $\pi^{\infty, \text{p}}$ to an automorphic representation π as in that proposition, such that $(\mathbf{V}_N^{\circ}, \pi)$ is a Π -congruent standard pair, and

that $\text{BC}(\pi_{\mathfrak{p}})$ is a constituent of an unramified principal series of $\text{GL}_N(F_{\mathfrak{p}})$, whose Satake parameter contains $-p$ and $-p^{-1}$ (which is then different from $\alpha(\Pi_{\mathfrak{p}})$ in $\overline{\mathbb{F}_{\ell}}$ by (P5)). On the other hand, the semisimplified residual representations of $\rho_{\text{BC}(\pi), \iota_{\ell}}$ and $\rho_{\Pi, \iota_{\ell}}$ are isomorphic. In particular, they have the same generalized Frobenius eigenvalues in $\overline{\mathbb{F}_{\ell}}$ at the unique place of F over $\mathfrak{p}$. However, this is not possible by Arthur's multiplicity formula (see [KMSW14, Theorem 1.7.1]), Proposition 2.1.1(3) and Proposition 3.2.2. Therefore, we must have $H_{\mathbb{F}}^{2r-1}(\overline{M}_N^{\dagger}, \mathcal{O}_{\lambda})_{\mathfrak{m}} = 0$. $\square$

Proposition 3.5.2. *Assume Assumption 3.4.2 and Hypothesis 3.2.3 for N .*

- (1) $E_{2, \mathfrak{m}}^{p, q}$ vanishes unless $(p, q) = (0, 2r)$, and $E_{2, \mathfrak{m}}^{0, 2r}$ is canonically isomorphic to $H_{\mathbb{F}}^{2r}(\overline{M}_N, R\Psi\mathcal{O}_{\lambda}(r))_{\mathfrak{m}}$, which is a free $\mathcal{O}_{\lambda}$ -module.
- (2) The set of generalized Frobenius eigenvalues of the $\kappa_{\lambda}[\text{Gal}_{\mathbb{F}_{p^2}}]$ -module $E_{2, \mathfrak{m}}^{0, 2r} \otimes_{\mathcal{O}_{\lambda}} \kappa_{\lambda}$ is contained in the set of roots of $P_{\alpha(\Pi_{\mathfrak{p}})}(\text{mod } \lambda)$ in $\overline{\mathbb{F}_{\ell}}$.
- (3) The $\mathcal{O}_{\lambda}[\text{Gal}_{\mathbb{F}_{p^2}}]$ -module $E_{2, \mathfrak{m}}^{0, 2r}$ is weakly semisimple.
- (4) The localization of the map ∇^1 at $\mathfrak{m}$ induces an isomorphism

$$\nabla_{\mathfrak{m}}^1 : \left(E_{2, \mathfrak{m}}^{0, 2r} \right)_{\text{Gal}_{\mathbb{F}_{p^2}}} \xrightarrow{\sim} \mathcal{O}_{\lambda}[\text{Sh}(\mathbf{V}_N^{\circ}, \mathcal{K}_N^{\circ})]_{\mathfrak{m}}.$$

Proof. For (1), by Lemma 3.5.1, the same proof of [LTX⁺22, Lemma 6.2.2(3)] goes through.

Next we prove parts (2)-(4). Firstly it follows from the proof of [LTX⁺22, Theorem 6.2.3] and [LTX24, Lemma 4.2.4] that $\nabla_{\mathfrak{m}}^1$ is surjective. By [LTX⁺22, Lemma 5.2.7] and part (1), there is an isomorphism

$$(3.3) \quad E_{2, \mathfrak{m}}^{0, 2r} \cong H_{\text{ét}}^{2r} \left(\text{Sh}(\mathbf{V}', j_N \mathcal{K}_N^{\circ, p} \mathcal{K}'_{N, p})_{\overline{F}}, \mathcal{O}_{\lambda}(r) \right)_{\mathfrak{m}}$$

of $\mathcal{O}_{\lambda}[\text{Gal}_{\mathbb{Q}_{p^2}}]$ -modules. By Lemma 3.4.6, 3.4.5, Hypothesis 3.2.3 and Arthur's multiplicity formula [KMSW14, Theorem 1.7.1], there is an isomorphism

$$(3.4) \quad H_{\text{ét}}^{2r} \left(\text{Sh}(\mathbf{V}', j_N \mathcal{K}_N^{\circ, p} \mathcal{K}'_{N, p})_{\overline{F}}, \mathcal{O}_{\lambda}(r) \right)_{\mathfrak{m}} \otimes_{\mathcal{O}_{\lambda}} \overline{\mathbb{Q}_{\ell}} \cong \bigoplus_{\pi'^{\infty}} (W^{N-1}(\pi'^{\infty}))^{\oplus d(\pi'^{\infty})}$$

of $\overline{\mathbb{Q}_{\ell}}[\text{Gal}_F]$ -modules, where $d(\pi'^{\infty}) = \dim(\pi'^{\infty})^{j_N \mathcal{K}_N^{\circ, p} \mathcal{K}'_{N, p}}$; and the sum is taken over all admissible irreducible representations π'^{∞} of $\text{U}(\mathbf{V}')(\mathbf{A}_{F_+}^{\infty})$ that is the finite part of some automorphic representation π' of $\text{U}(\mathbf{V}')(\mathbf{A}_{F_+})$ satisfying $(\mathbf{V}', \pi')$ is a standard pair. Here we choose such a π' for each π'^{∞} appearing in the direct sum. For the proof of parts (2-4), we may replace E_{λ} by a finite extension inside $\overline{\mathbb{Q}_{\ell}}$ such that $W^{N-1}(\pi'^{\infty})$ is defined over E_{λ} for each π'^{∞} appearing in the previous direct sum. For each such π'^{∞} , $W^{N-1}(\pi'^{\infty})$ is conjugate to an irreducible subrepresentation of $\rho_{\Pi, \iota_{\ell}}^c$ by Hypothesis 3.2.3. Thus part (2) follows from Equations (3.3), (3.4) and part (1).

For (3), we choose a Gal_F -stable $\mathcal{O}_{\lambda}$ -lattice $R^{N-1}(\pi'^{\infty})$ of $W^{N-1}(\pi'^{\infty})$ for each π'^{∞} appearing in the previous direct sum. We claim that $R^{N-1}(\pi'^{\infty})$ is weakly semisimple, which implies part (3) by [LTX⁺22, Lemma 2.1.4(1)]. By (P4), we know $\bar{\rho}_{\Pi, \lambda}^c(r)$ is weakly semisimple, and

$$\dim_{\kappa_{\lambda}} \bar{\rho}_{\Pi, \lambda}(r)^{\text{Gal}_{\mathbb{F}_{p^2}}} = 0, \quad \dim_{\kappa_{\lambda}} \bar{\rho}_{\Pi^{\flat}, \lambda}(r)^{\text{Gal}_{\mathbb{F}_{p^2}}} = 1.$$

If $\dim_{E_{\lambda}} W^{N-1}(\pi'^{\infty})$ is odd, then

$$\dim_{E_{\lambda}} W^{N-1}(\pi'^{\infty})^{\text{Gal}_{\mathbb{F}_{p^2}}} \geq 1.$$

Thus $R^{N-1}(\pi'^{\infty})$ is weakly semisimple by [LTX⁺22, Lemma 2.1.5]. On the other hand, if $\dim_{E_{\lambda}} W^{N-1}(\pi'^{\infty})$ is even, then $\text{BC}(\pi')$ is almost cuspidal, and $R^{N-1}(\pi'^{\infty}) \otimes_{\mathcal{O}_{\lambda}} \kappa_{\lambda}$ is conjugate to $\rho_{\Pi^{\flat}, \lambda}$ as $\kappa_{\lambda}[\text{Gal}_F]$ -modules. Thus $R^{N-1}(\pi'^{\infty})$ is also weakly semisimple by [LTX⁺22, Lemma 2.1.5].

For (4): By the above discussion, it suffices to show

$$\sum_{\pi'^{\infty}} d(\pi'^{\infty}) \leq \dim_{E_{\lambda}} \mathcal{O}_{\lambda}[\text{Sh}(\mathbf{V}_N^{\circ}, \mathcal{K}_N^{\circ})]_{\mathfrak{m}} \otimes_{\mathcal{O}_{\lambda}} E_{\lambda},$$

where π'^{∞} is taken over all those appearing in the previous direct sum satisfying $\dim_{E_{\lambda}} W^{N-1}(\pi'^{\infty})$ is odd. This assertion follows from Lemma 3.4.7 and Lemma 3.5.3 below. $\square$

Lemma 3.5.3. *Let π' be an automorphic representation of $U(\mathbf{V}')(\mathbf{A}_{F_+})$ such that $BC(\pi')$ is relevant. If N is odd and $BC(\pi') = \Pi^{\mathfrak{p}} \boxplus \chi$ is almost cuspidal, we further assume the following conditions.*

- *The local component $\pi'_{\mathfrak{p}}$ is unramified with Satake parameter containing 1 exactly once.*
- *Set $\mathfrak{I} = \{N-1, N-3, \dots, 3-N, 1-N\}$, in particular $\chi_{\tau_{\infty}}(z) = \arg(z)^{a_{\chi}}$ for some $a_{\chi} \in \mathfrak{I}$. Let $\kappa_{\chi} : \mu_2^{\mathfrak{I}} \rightarrow \mathbb{C}^{\times}$ denote the character that takes value -1 on the generator corresponding to $a_{\chi} \in \mathfrak{I}$ and takes value 1 on all other generators. Then $\pi'_{\tau_{\infty}}$ is isomorphic to the discrete series $\pi^{\kappa_{\chi}}$ (with Harish-Chandra parameter $(r, r-1, \dots, 1-r, -r)$) as defined in [LL21, Notation 3.14].*

Consider the admissible irreducible representation $\pi := \pi_{\tau_{\infty}} \otimes \pi_{\mathfrak{p}} \otimes (\pi')^{\tau_{\infty}, \mathfrak{p}}$ of $U(\mathbf{V}^{\circ})(\mathbf{A}_{F_+})$ where

- *$\pi_{\tau_{\infty}}$ is the trivial representation of $U(\mathbf{V}^{\circ} \otimes_F F_{\tau_{\infty}})$;*
- *$\pi_{\mathfrak{p}}$ is an unramified representation of $U(\mathbf{V}^{\circ} \otimes_F F_{\mathfrak{p}})$ satisfying $BC(\pi_{\mathfrak{p}}) = BC(\pi'_{\mathfrak{p}})$.*

Then the automorphic multiplicity of π is 1.

Proof. This follows from Arthur's multiplicity formula for tempered global L-packets; cf. [KMSW14, Theorem 1.7.1]. $\square$

3.6. Arithmetic level-raising in the even rank case. In this subsection, we assume that N is even and work in the setting of Setup 3.4.1.

We recall the following definition of rigid residual Galois representations from [LTX+24, §3.6].

Definition 3.6.1. Let $\bar{r} : \text{Gal}_{F_+} \rightarrow \mathcal{G}_N(\kappa_{\lambda})$ be a continuous homomorphism satisfying

$$\bar{r}^{-1}(\text{GL}_N(\kappa_{\lambda}) \times \text{GL}_1(\kappa_{\lambda})) = \text{Gal}_F, \quad \nu \circ \bar{r} = \eta_{F/F_+}^N \bar{\varepsilon}_{\ell}^{1-N}.$$

We say $\bar{r}$ is rigid for $(\Sigma_+^{\min}, \Sigma_+^{\text{lr}})$ if the following are satisfied:

- (1) For $v \in \Sigma_+^{\min}$, any lifting $r : \text{Gal}_{F_+} \rightarrow \mathcal{G}_N(\kappa_{\lambda})$ with $\nu \circ r = \eta_{F/F_+}^N \varepsilon_{\ell}^{1-N}$ is minimally ramified as defined in [LTX+24, Definition 3.4.8].
- (2) For $v \in \Sigma_+^{\text{lr}}$, the set of generalized eigenvalues of $\bar{r}_v^{\sharp}(\phi_w)$ contains the pair $\{\|v\|^{-N}, \|v\|^{-N+2}\}$ exactly once, where w is the unique place of F over v .
- (3) For $v \in \Sigma_{F_+}(\ell)$, $\bar{r}_v^{\sharp}$ is regular Fontaine–Laffaille crystalline as defined in [LTX+24, Definition 3.2.4].
- (4) For a finite place of F_+ not in $\Sigma_+^{\min} \cup \Sigma_+^{\text{lr}} \cup \Sigma_{F_+}(\ell)$, $\bar{r}_v$ is unramified.

We state the following variant of the R=T theorem in [LTX+24] suitable for our case. We apply the discussion of [LTX+24, §3] to the pair $(\bar{r}, \chi) = (\bar{r}_{\Pi, \lambda, +}, \varepsilon_{\ell}^{1-N})$. Suppose $\bar{r}$ is rigid for $(\Sigma_+^{\min}, \Sigma_+^{\text{lr}})$. For each $? \in \{\text{mix}, \text{unr}, \text{ram}\}$, we consider the global deformation problem (see [LTX+24, Definition 3.6])

$$\mathcal{S}^? = \left(\bar{r}, \varepsilon_{\ell}^{1-N}, \Sigma_+^{\min} \cup \Sigma_+^{\text{lr}} \cup \{\mathfrak{p}\} \cup \Sigma_{F_+}(\ell), \{\mathcal{D}_v\}_{v \in \Sigma_+^{\min} \cup \Sigma_+^{\text{lr}} \cup \{\mathfrak{p}\} \cup \Sigma_{F_+}(\ell)} \right)$$

where

- for $v \in \Sigma_+^{\min}$, $\mathcal{D}_v$ is the local deformation problem classifying all liftings of $\bar{r}_v$;
- for $v \in \Sigma_+^{\text{lr}}$, $\mathcal{D}_v$ is the local deformation problem $\mathcal{D}_v^{\text{ram}}$ of $\bar{r}_v$ from [LTX+24, Definition 3.34];
- for $v = \mathfrak{p}$, $\mathcal{D}_v$ is the local deformation problem $\mathcal{D}^?$ of $\bar{r}_v$ from [LTX+24, Definition 3.34];
- for $v \in \Sigma_{F_+}(\ell)$, $\mathcal{D}_v$ is the local deformation problem $\mathcal{D}_v^{\text{FL}}$ of $\bar{r}_v$ from [LTX+24, Definition 3.12].

Then the global universal deformation ring $R_{\mathcal{S}^?}^{\text{univ}}$ is defined in [LTX+24, Proposition 3.7]. Set $R^? := R_{\mathcal{S}^?}^{\text{univ}}$ for short. Then there are canonical surjective homomorphisms $R^{\text{mix}} \rightarrow R^{\text{unr}}$ and $R^{\text{mix}} \rightarrow R^{\text{ram}}$ of $\mathcal{O}_{\lambda}$ -rings. We have the following corollary of the R = T theorem from [LTX+24].

Theorem 3.6.2. *Assume Assumptions 3.4.2 and Hypothesis 3.2.3 for N . We further assume that $\ell \geq 2(N+1)$, $\bar{r}_{\Pi, \lambda, +}$ (Equation (3.2)) is rigid for $(\Sigma_+^{\min}, \Sigma_+^{\text{lr}})$, and $\bar{r}_{\Pi, \lambda}|_{\text{Gal}_{F(\mu_{\ell})}}$ is absolutely irreducible.*

- *Let T^{unr} denote the image of $\mathbb{T}_N^{\Sigma_+ \cup \{\mathfrak{p}\}}$ in $\text{End}_{\mathcal{O}_{\lambda}}(\mathcal{O}_{\lambda}[\text{Sh}(\mathbf{V}_N^{\circ}, \mathcal{K}_N^{\circ})])$. Then there is a canonical isomorphism $R^{\text{unr}} \cong T^{\text{unr}}$ of nonzero local rings such that $\mathcal{O}_{\lambda}[\text{Sh}(\mathbf{V}_N^{\circ}, \mathcal{K}_N^{\circ})]$ is a nonzero finite free module over R^{unr} .*
- *Let T^{ram} denote the image of $\mathbb{T}_N^{\Sigma_+ \cup \{\mathfrak{p}\}}$ in $\text{End}_{\mathcal{O}_{\lambda}}(H_{\Sigma}^{2r-1}(\overline{\mathbf{M}}_N, R\Psi\mathcal{O}_{\lambda}))$. Then there is a canonical isomorphism $R^{\text{ram}} \cong T^{\text{ram}}$ of nonzero local rings such that $\mathcal{O}_{\lambda}[\text{Sh}(\mathbf{V}_N^{\circ}, \mathcal{K}_N^{\circ})]$ is a nonzero finite free module over R^{ram} .*

Proof. For (1): This follows from [LTX⁺24, Theorem 3.38], except that when $v \in \Sigma_{F_+}^{\text{fin}} \setminus \Sigma_+^{\text{min}}$, the level group $(\mathcal{K}_N^\circ)_v$ is a hyperspecial but may not be the stabilizer of a self-dual lattice in $U(\mathbf{V}_N^\circ)(F_{+,v})$. However, the proof of [LTX⁺24, Theorem 3.38] goes through.

For (2): By [LTX⁺22, Proposition 3.6.1], we know $\mathbf{T}^{\text{ram}}$ is nonzero. Thus by [LTX⁺22, Lemma 5.2.7] and the same reason as in (1), the assertion follows from [LTX⁺24, Theorem 3.38] (with $(\Sigma_+^{\text{min}}, \Sigma_+^{\text{lr}})$ replaced by $(\Sigma_+^{\text{min}}, \Sigma_+^{\text{lr}} \cup \{\mathfrak{p}\})$). $\square$

Proposition 3.6.3. *Assume Assumptions 3.4.2 and Hypothesis 3.2.3 for N . Assume further that $\ell \geq 2(N+1)$, $\bar{\rho}_{\Pi, \lambda, +}$ (Equation (3.2)) is rigid for $(\Sigma_+^{\text{min}}, \Sigma_+^{\text{lr}})$, and $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}_F(\mu_\ell)}$ is absolutely irreducible.*

- (1) $H_{\Sigma}^i(\bar{M}_N^\bullet, \mathcal{O}_\lambda)_{\mathfrak{m}}$ is a free $\mathcal{O}_\lambda$ -module for every $i \in \mathbb{Z}_+$.
- (2) $E_{2, \mathfrak{m}}^{p, q}$ is a free $\mathcal{O}_\lambda$ -module, and vanishes unless $p+q=2r-1$ and $|p| \leq 1$.
- (3) The set of generalized Frobenius eigenvalues of the $\kappa_\lambda[\text{Gal}_{\mathbb{F}_{p^2}}]$ -module $H_{\Sigma}^{2r-1}(\bar{M}_N^\bullet, \mathcal{O}_\lambda(r))_{\mathfrak{m}} \otimes_{\mathcal{O}_\lambda} \kappa_\lambda$ is contained in the set of roots of $P_{\alpha(\Pi_0, \mathfrak{p})}(p^{-1}T) \pmod{\lambda}$ in $\mathbb{F}_\ell$, and does not contain 1 or p^2 .
- (4) The quotient modulo $\mathfrak{n}$ of the map $\nabla_{\mathfrak{n}}^0$ induces an isomorphism

$$\nabla_{\mathfrak{n}}^0 : F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\Sigma}^{2r-1}(\bar{M}_N, R\Psi\mathcal{O}_\lambda(r)) / \mathfrak{n}) \xrightarrow{\sim} \mathcal{O}_\lambda[\text{Sh}(\mathbf{V}^\circ, \mathcal{K}_N^\circ)] / \mathfrak{n}.$$

Here F_{-1} is the degree -1 term of the monodromy filtration.

- (5) There is a natural isomorphism

$$F_{-1}H^1(I_{\mathbb{Q}_{p^2}}, H_{\text{ét}}^{2r-1}(\bar{M}_N, R\Psi\mathcal{O}_\lambda(r)) / \mathfrak{n}) \cong H_{\text{sing}}^1(\mathbb{Q}_{p^2}, H_{\text{ét}}^{2r-1}(\bar{M}_N, R\Psi\mathcal{O}_\lambda(r)) / \mathfrak{n}).$$

- (6) There exists a positive integer $\mu \in \mathbb{Z}_+$ and an isomorphism

$$H_{\text{ét}}^{2r-1}(\text{Sh}(\mathbf{V}'_N, j_N \mathcal{K}_N^{\infty, p} \mathcal{K}'_{N, p})_{\bar{F}}, \mathcal{O}_\lambda(r)) / \mathfrak{n} \cong (\bar{R}^{(m)^c})^{\oplus \mu}$$

of $\mathcal{O}_\lambda[\text{Gal}_F]$ -modules, where R is a Gal_F -stable $\mathcal{O}_\lambda$ -lattice in $\rho_{\Pi, \lambda}(r)$, unique up to homothety.

Proof. For (1)-(3): Using Theorem 3.6.2, the proof of [LTX⁺22, Theorem 6.3.4(1)-(3)] goes through.

For (6)-(7): By the proof of [LTX⁺22, Theorem 6.3.4(4)], these follow from [LTX⁺22, Proposition 6.4.1], [LTX24, Lemma 4.2.4(2)] and Theorem 3.6.2. $\square$

3.7. First explicit reciprocity law. We now work in the following setup.

Setup 3.7.1.

- Let $n \geq 2$ be an integer. Among $\{n, n+1\}$, $n_0 = 2r_0$ (resp. $n_1 = 2r_1 + 1$) be the unique even (resp. odd) number in the set $\{n, n+1\}$. In particular, $r_0 + r_1 = n$.
- Let Π_0 be a cuspidal relevant representation of $\text{GL}_{n_0}(\mathbf{A}_F)$, and let Π_1 be an almost cuspidal relevant representation of $\text{GL}_{n_1}(\mathbf{A}_F)$ of the form $\Pi_1 = \Pi_1^\flat \boxplus \mathbf{1}$, where $\mathbf{1}$ is the trivial character of $\text{GL}_1(\mathbf{A}_F)$ (see Definition 1.1.3).
- Let $E \subset \mathbb{C}$ be a strong coefficient field of Π (see Definition 3.1.6).
- For each $\alpha \in \{0, 1\}$ and each finite place λ of E , let $\rho_{\Pi_\alpha, \lambda} : \text{Gal}_F \rightarrow \text{GL}_{n_\alpha}(E_\lambda)$ be the continuous homomorphism attached to Π_α (see Definition 3.1.6). In particular, $\rho_{\Pi_\alpha, \lambda}^c$ and $\rho_{\Pi_\alpha, \lambda}^\vee(1 - n_\alpha)$ are conjugate.
- For each $\alpha \in \{0, 1\}$, let $\phi_{\Pi_\alpha} : \mathbb{T}_{n_\alpha}^{\Sigma_{+}^{\Pi_0} \cup \Sigma_{+}^{\Pi_1}} \rightarrow \mathcal{O}_E$ be the restriction of the Hecke character defined in Definition 3.1.9.

We further assume that we are in the following setting.

Setup 3.7.2. Let $(\lambda, \Sigma_+^{\text{lr}, I}, \Sigma_+^I, \mathcal{V}^\circ, m, \mathfrak{p}, \mathcal{T}, \mathcal{V}^\bullet, \mathcal{U})$ be a nonuple, where

- λ is a finite place of E whose underlying prime ℓ satisfies $\Sigma_+^{\Pi_0} \cap \Sigma_{F_+}(\ell) = \emptyset$ and $\ell \geq 2(n_0 + 1)$.
- $\Sigma_+^{\text{lr}, I}$ is a finite set of finite inert places of F_+ strongly disjoint from $\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}$ (see Definition 3.1.1) satisfying $\ell \nmid \|v\|(\|v\|^2 - 1)$ for any $v \in \Sigma_+^{\text{lr}, I}$.
- Σ_+^I is a finite set of finite places of F_+ containing $\Sigma_+^{\text{lr}, I}$ and $\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}$.

- $\mathcal{V}^\circ = (\mathbf{V}_n^\circ, \mathbf{V}_{n+1}^\circ; \Lambda_n^\circ, \Lambda_{n+1}^\circ; \mathcal{K}_n^\circ, \mathcal{K}_{\text{sp}}^\circ, \mathcal{K}_{n+1}^\circ)$ is a septuple, where⁵
 - (1) $\mathbf{V}_n^\circ$ is a standard definite Hermitian space over F of dimension N (see Definition 3.2.1), and $\mathbf{V}_{n+1}^\circ = (\mathbf{V}_n^\circ)_\#$, such that $(\mathbf{V}_{n_0}^\circ)_v$ is not split for $v \in \Sigma_+^{\text{lr}, \text{I}}$.
 - (2) Λ_n° is a self-dual $\prod_{v \in \Sigma_F^\infty \setminus \Sigma_+^{\text{I}}} \mathcal{O}_{F_v}$ -lattice in $\mathbf{V}_n^\circ \otimes_{F_+} \mathbf{A}_{F_+}^{\infty, \Sigma_+^{\text{I}}}$;
 - (3) $\mathcal{K}_n^\circ$ is an object in $\mathfrak{K}(\mathbf{V}_n^\circ)$ and $(\mathcal{K}_{\text{sp}}^\circ, \mathcal{K}_{n+1}^\circ)$ is an object in $\mathfrak{K}(\mathbf{V}_n^\circ)_{\text{sp}}$ of the forms

$$\begin{aligned}\mathcal{K}_n^\circ &= \prod_{v \in \Sigma_+^{\text{I}}} (\mathcal{K}_n^\circ)_v \times \prod_{v \in \Sigma_+^{\text{fin}} \setminus \Sigma_+^{\text{I}}} \text{U}(\Lambda_n^\circ)(\mathcal{O}_v), \\ \mathcal{K}_{\text{sp}}^\circ &= \prod_{v \in \Sigma_+^{\text{I}}} (\mathcal{K}_{\text{sp}}^\circ)_v \times \prod_{v \in \Sigma_+^{\text{fin}} \setminus \Sigma_+^{\text{I}}} \text{U}(\Lambda_n^\circ)(\mathcal{O}_v), \\ \mathcal{K}_{n+1}^\circ &= \prod_{v \in \Sigma_+^{\text{I}}} (\mathcal{K}_{n+1}^\circ)_v \times \prod_{v \in \Sigma_+^{\text{fin}} \setminus \Sigma_+^{\text{I}}} \text{U}(\Lambda_{n+1}^\circ)(\mathcal{O}_v),\end{aligned}$$

satisfying

- $(\mathcal{K}_{\text{sp}}^\circ)_v \subset (\mathcal{K}_n^\circ)_v$ for $v \in \Sigma_+^{\text{I}}$, and
- $(\mathcal{K}_{n_0}^\circ)_v$ is a hyperspecial maximal subgroup of $\text{U}(\mathbf{V}_{n_0}^\circ)(F_v)$ for $v \in \Sigma_+^{\text{I}} \setminus (\Sigma_+^{\text{lr}, \text{I}} \cup \Sigma_+^{\text{II}_0})$, and is a special maximal subgroup of $\text{U}(\mathbf{V}_{n_0}^\circ)(F_v)$ for $v \in \Sigma_+^{\text{lr}, \text{I}}$

such that

$$\frac{\mathcal{O}_\lambda[\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ)]}{\mathbb{T}_{n_0}^{\Sigma_+^{\text{I}}} \cap \ker \phi_{\Pi_{n_0}}}$$

is nontrivial.

- $m \in \mathbb{Z}_+$ is a positive integer,
- $\mathfrak{p}$ is a very good inert place of F_+ with the underlying rational prime p (see Definition 3.3.3), satisfying⁶
 - (PI1) $\mathfrak{p}$ is strongly disjoint from Σ_+^{I} ;
 - (PI2) ℓ does not divide $p(p^2 - 1)$;
 - (PI3) There exists a CM type Φ containing τ_∞ as in [LTX⁺22, §5.1] with $\mathbb{Q}_{p^2}^\Phi = \mathbb{Q}_{p^2}$ (we refer to [LTX⁺22, §3.3] for the definitions).
 - (PI4) $P_{\alpha(\Pi_{0,\mathfrak{p}})}(\text{mod } \lambda^m)$ is level-raising special at $\mathfrak{p}$, $P_{\alpha(\Pi_{1,\mathfrak{p}})}(\text{mod } \lambda)$ is Tate generic at $\mathfrak{p}$, and $P_{\alpha(\Pi_{0,\mathfrak{p}}) \otimes \alpha(\Pi_{1,\mathfrak{p}})}(\text{mod } \lambda^m)$ is level-raising special at $\mathfrak{p}$ (see Definition 3.1.4);
 - (PI5) $P_{\alpha(\Pi_{\alpha,\mathfrak{p}})}(\text{mod } \lambda)$ is intertwining generic at $\mathfrak{p}$ for each $\alpha \in \{0, 1\}$.

In particular, we can and will apply the construction and notations in [LTX⁺22, §5.10] to the datum $(\mathbf{V}_n^\circ, \{\Lambda_{n,q}^\circ\}_{q|p})$. For each $\alpha \in \{0, 1\}$, denote by

$$\mathfrak{m}_\alpha := \mathbb{T}_{n_\alpha}^{\Sigma_+^{\text{I}} \cup \Sigma_{F_+}(p)} \cap \ker \left(\mathbb{T}_{n_\alpha}^{\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}} \xrightarrow{\phi_\Pi} \mathcal{O}_E \rightarrow \mathcal{O}_E/\lambda \right)$$

and

$$\mathfrak{n}_\alpha := \mathbb{T}_{n_\alpha}^{\Sigma_+^{\text{I}} \cup \Sigma_{F_+}(p)} \cap \ker \left(\mathbb{T}_{n_\alpha}^{\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}} \xrightarrow{\phi_\Pi} \mathcal{O}_E \rightarrow \mathcal{O}_E/\lambda^m \right)$$

the two ideals of $\mathbb{T}_{n_\alpha}^{\Sigma_+^{\text{I}} \cup \Sigma_{F_+}(p)}$.

- $\mathcal{T} = (\Phi, \mathbf{W}_0, \mathcal{K}_0^p, \iota_p, \varpi)$ is a quintuple of data as in [LTX⁺22, §5.1] with $\mathbb{Q}_p^\Phi = \mathbb{Q}_{p^2}$.
- $\mathcal{V}^\bullet = (\Lambda_{n,\mathfrak{p}}^\bullet, \Lambda_{n+1,\mathfrak{p}}^\bullet; \mathcal{K}_{n,p}^\bullet, \mathcal{K}_{n+1,p}^\bullet, \mathcal{K}_{\text{sp},p}^\bullet; \mathcal{K}_{n,p}^\dagger, \mathcal{K}_{\text{sp},p}^\dagger, \mathcal{K}_{n+1,p}^\dagger)$ is an octuple of data as in [LTX⁺22, Notation 5.10.13]. For each $\alpha \in \{0, 1\}$, we set $\mathcal{K}_{n_\alpha}^{\circ,p} := (\mathcal{K}_{n_\alpha}^\circ)^p$, and $\mathcal{K}_{n_\alpha}^\bullet := \mathcal{K}_{n_\alpha}^{\circ,p} \times \mathcal{K}_{n_\alpha,p}^\bullet$.

⁵Compared with [LTX⁺22, §7.2], we omit the assumption that $(\mathcal{K}_N^\circ)_v$ is transferable when N is even, which is possible by [LTX⁺25, Remark 8.2]

⁶Compared with [LTX⁺22, §7.2], we incorporate (PI7) into (PI4), and omit assumption (PI6) as it will be redundant for applications in view of [LTX24, Lemma 4.2.4(2)].

- $\mathcal{U} = (\mathbf{V}'_n, j_n, \{\Lambda'_{n,q}\}_{q|p}; \mathbf{V}'_{n+1}, j_{n+1}, \{\Lambda'_{n+1,q}\}_{q|p})$ is a sextuple in which $(\mathbf{V}'_n, j_n, \{\Lambda'_{n,q}\}_{q|p})$ is an indefinite uniformization datum for $\mathbf{V}'_n$ as in Setup 3.4.1, $\mathbf{V}'_{n+1} := (\mathbf{V}'_n)_\#$, $j_{n+1} := (j_n)_\#$, and $\Lambda_{n+1,q} = (\Lambda_{n,q})_\#$ for each $q|p$. Then $(\mathbf{V}'_{n+1}, j_{n+1}, \{\Lambda'_{n+1,q}\}_{q|p})$ is an indefinite uniformization datum for $\mathbf{V}'_{n+1}$. For each $\alpha \in \{0, 1\}$, let $\mathcal{K}'_{n_\alpha, q}$ denote the stabilizer of $\Lambda'_{n_\alpha, q}$, and set $\mathcal{K}'_{n_\alpha, p} := \prod_{q|p} \mathcal{K}'_{n_\alpha, q}$.

For each $\alpha \in \{0, 1\}$, we set $X_{n_\alpha}^? := X_{\mathbf{p}}^?(\mathbf{V}_{n_\alpha}^\circ, \mathcal{K}_{n_\alpha}^{p, \circ})$ for meaningful pairs $(X, ?) \in \{\mathbf{M}, \mathbf{M}, \mathbf{B}, \mathbf{S}\} \times \{, \eta, \circ, \bullet, \dagger\}$, and let $({}^\alpha E_s^{p, q}, {}^\alpha d_s^{p, q})$ denote the weight spectral sequence abutting to the cohomology $H_{\mathfrak{T}}^\bullet(\overline{\mathbf{M}}_{n_\alpha}, R\Psi\mathcal{O}_\lambda(r_\alpha))$ from [LTX+22, §5.9].

Assumption 3.7.3. $\rho_{\Pi_0, \lambda}$ and $\rho_{\Pi_1, \lambda}$ are residually absolutely irreducible.

Under Assumption 3.7.3, for each $\alpha \in \{0, 1\}$, we get a residual representation $\bar{\rho}_{\Pi_\alpha, \lambda}$, which is unique up to conjugation and $(1 - n_\alpha)$ -polarizable in the sense of [LTX+22, Definition 2.5.3]. Then we obtain a continuous homomorphism

$$(3.5) \quad \bar{\rho}_{\Pi_\alpha, \lambda, +} : \text{Gal}_{F_+} \rightarrow \mathcal{G}_{n_\alpha}(\kappa_\lambda)$$

from [LTX+22, Construction 2.5.4].

Assumption 3.7.4. Assumption 3.7.3 holds, $\bar{\rho}_{\Pi_0, \lambda, +}$ is rigid for $(\Sigma_+^{\Pi_0}, \Sigma_+^{\text{lr}, \text{I}})$ (see Definition 3.6.1), and $\bar{\rho}_{\Pi_0, \lambda}|_{\text{Gal}_{F(\mu_\ell)}}$ is absolutely irreducible.

Assumption 3.7.5. For each $\alpha \in \{0, 1\}$, the composite homomorphisms $\mathbb{T}_{n_\alpha}^{\Sigma^{\text{min}} +} \xrightarrow{\phi_{\Pi_\alpha}} \mathcal{O}_E \rightarrow \kappa_\lambda$ is cohomologically generic (see Definition 3.2.5).

In the following we will freely use the notation from [LTX+22, §7.2].

We apply the construction and notation of [LTX+22, §5.11], evaluating on the object $(\mathcal{K}_n^{\circ, p}, \mathcal{K}_{n+1}^{\circ, p}) \in \mathfrak{K}(\mathbf{V}_n^\circ)^p \times \mathfrak{K}(\mathbf{V}_{n+1}^\circ)^p$. In particular, we obtain the blow-up morphism $\sigma : \mathbf{Q} \rightarrow \mathbf{P}$ from [LTX+22, Notation 5.11.1], and the localized weight spectral sequence $(\mathbb{E}_{s, (\mathbf{m}_0, \mathbf{m}_1)}^{p, q}, d_{s, (\mathbf{m}_0, \mathbf{m}_1)}^{p, q})$ abutting to the cohomology $H_{\mathfrak{T}}^\bullet(\overline{\mathbf{Q}}, R\Psi\mathcal{O}_\lambda(n))_{(\mathbf{m}_0, \mathbf{m}_1)}$ from [LTX+22, (5.27)].

Lemma 3.7.6. Assume Assumptions 3.7.4, 3.7.5 and Hypothesis 3.2.3 for each $N \in \{n, n+1\}$. Then

(1) For any $(?_0, ?_1) \in \{\circ, \bullet, \dagger\}^2$ and any $i \in \mathbb{Z}$, there is a canonical isomorphism

$$H_{\mathfrak{T}}^i(\overline{\mathbf{P}}^{?_0, ?_1}, \mathcal{O}_\lambda(i))_{(\mathbf{m}_0, \mathbf{m}_1)} \cong \bigoplus_{i_0 + i_1 = i} H_{\mathfrak{T}}^{i_0}(\overline{\mathbf{M}}_{n_0}^{?_0}, \mathcal{O}_\lambda)_{\mathbf{m}_0} \otimes_{\mathcal{O}_\lambda} H_{\mathfrak{T}}^{i_1}(\overline{\mathbf{M}}_{n_1}^{?_1}, \mathcal{O}_\lambda)_{\mathbf{m}_1}$$

in $\text{Mod}(\text{Gal}_{\mathbb{F}_{p^2}}, \mathcal{O}_\lambda)_{\text{fr}}$.

(2) $\mathbb{E}_{2, (\mathbf{m}_0, \mathbf{m}_1)}^{p, q}$ vanishes unless $(p, q) \in \{(-1, 2n), (0, 2n-1), (1, 2n-2)\}$, and canonical isomorphisms

$$\begin{cases} \mathbb{E}_{2, (\mathbf{m}_0, \mathbf{m}_1)}^{-1, 2n} \cong {}^0 E_{2, \mathbf{m}_0}^{-1, 2r_0} \otimes_{\mathcal{O}_\lambda} {}^1 E_{2, \mathbf{m}_1}^{0, 2r_1}, \\ \mathbb{E}_{2, (\mathbf{m}_0, \mathbf{m}_1)}^{0, 2n-1} \cong {}^0 E_{2, \mathbf{m}_0}^{0, 2r_0-1} \otimes_{\mathcal{O}_\lambda} {}^1 E_{2, \mathbf{m}_1}^{0, 2r_1}, \\ \mathbb{E}_{2, (\mathbf{m}_0, \mathbf{m}_1)}^{1, 2n-2} \cong {}^0 E_{2, \mathbf{m}_0}^{1, 2r_0-2} \otimes_{\mathcal{O}_\lambda} {}^1 E_{2, \mathbf{m}_1}^{0, 2r_1}, \end{cases}$$

in $\text{Mod}(\text{Gal}_{\mathbb{F}_{p^2}}, \mathcal{O}_\lambda)_{\text{lr}}$. In particular, $H_{\mathfrak{T}}^i(\overline{\mathbf{Q}}, R\Psi\mathcal{O}_\lambda(n))_{(\mathbf{m}_0, \mathbf{m}_1)}$ vanishes unless $i = 2n - 1$.

(3) If $\mathbb{E}_{2, (\mathbf{m}_0, \mathbf{m}_1)}^{i, 2n-1-i}(-1)$ has a nontrivial subquotient on which $\text{Gal}_{\mathbb{F}_{p^2}}$ acts trivially, then $i = 1$.

(4) For any $(?_0, ?_1) \in \{\circ, \bullet, \dagger\}^2$ and any $i \in \mathbb{Z}$, $H_{\mathfrak{T}}^{2i}(\overline{\mathbf{Q}}^{?_0, ?_1}, \mathcal{O}_\lambda(i))_{(\mathbf{m}_0, \mathbf{m}_1)}$ is weakly semisimple.

(5) The canonical map $H_{\mathfrak{T}, c}^i(\overline{\mathbf{Q}}^{(c)}, \mathcal{O}_\lambda)_{(\mathbf{m}_0, \mathbf{m}_1)} \rightarrow H_{\mathfrak{T}}^i(\overline{\mathbf{Q}}^{(c)}, \mathcal{O}_\lambda)_{(\mathbf{m}_0, \mathbf{m}_1)}$ is an isomorphism for any integers c and i .

Proof. For (1), By [LTX+22, Lemma 5.6.2], Lemma 3.5.2(1) and Lemma 3.6.3(1), we know that $H_{\mathfrak{T}}^{i_\alpha}(\overline{\mathbf{M}}_{n_\alpha}^{?_\alpha}, \mathcal{O}_\lambda)_{\mathbf{m}_\alpha}$ is a free $\mathcal{O}_\lambda$ -module for every $(\alpha, i_\alpha, ?_\alpha) \in \{0, 1\} \times \mathbb{N} \times \{\circ, \bullet, \dagger\}$. Thus (1) follows from Lemma 3.4.5 and the Künneth formula.

For (2), Using Lemma 3.5.1, Propositions 3.5.2, 3.6.3(2) and Lemma 3.4.5, the proof of [LTX+22, Lemma 7.2.5(2)] goes through.

For (3), by inspecting the proof of [LTX+22, Lemma 7.2.5(3)], the assertion follows from Proposition 3.5.2(2) and Proposition 3.6.3(3).

For (4): Using Proposition 3.5.2, the proof of [LTX+22, Lemma 7.2.5(4)] goes through.

Part (5) follows from part (1), Lemma 3.4.5 and [LTX+22, Lemma 5.11.3(3-5)]. $\square$

By Lemma 3.7.6(2), we obtain a coboundary map

$$\mathrm{AJ}_{\mathbf{Q}} : Z_{\mathbb{F}}^n(\mathbf{Q}^\eta) \rightarrow H^1 \left(\mathbb{Q}_{p^2}, H_{\mathbb{F}}^{2n-1}(\overline{\mathbf{Q}}, R\Psi\mathcal{O}_\lambda(n))_{(\mathfrak{m}_0, \mathfrak{m}_1)} \right).$$

By our choice of $\mathcal{K}_n^\circ$ and $(\mathcal{K}_{\mathrm{sp}}^\circ, \mathcal{K}_{n+1}^\circ)$, we obtain a finite morphism

$$\mathbf{M}_{\mathbf{P}}(\mathbf{V}_n^\circ, \mathcal{K}_{\mathrm{sp}}^\circ) \rightarrow \mathbf{P}.$$

Denote by $\mathbf{P}_{\mathrm{sp}}$ the corresponding cycle, and by $\mathbf{Q}_{\mathrm{sp}}$ the strict transform of $\mathbf{P}_{\mathrm{sp}}$ under σ , and $\mathbf{Q}_{\mathrm{sp}}$ the special fiber of $\mathbf{Q}_{\mathrm{sp}}$.

We recall the construction of potential map from [LTX+22, §5.11]. For each $r \in \mathbb{Z}$, set

$$B^r(\mathbf{Q}, \mathcal{O}_\lambda) := \ker \left(\delta_0^* : H_{\mathbb{F}}^{2r}(\overline{\mathbf{Q}}^{(0)}, \mathcal{O}_\lambda(r)) \rightarrow H_{\mathbb{F}}^{2r}(\overline{\mathbf{Q}}^{(1)}, \mathcal{O}_\lambda(r)) \right),$$

and

$$\begin{aligned} B_{n-r}(\mathbf{Q}, \mathcal{O}_\lambda) &:= \mathrm{Coker} \left(\delta_{1,!} : H_{\mathbb{F}}^{2(n+r-2)}(\overline{\mathbf{Q}}^{(1)}, \mathcal{O}_\lambda(n+r-2)) \right. \\ &\quad \left. \rightarrow H_{\mathbb{F}}^{2(n+r-1)}(\overline{\mathbf{Q}}^{(0)}, \mathcal{O}_\lambda(n+r-1)) \right), \end{aligned}$$

where δ_0^* is a linear combination of pullback maps and $\delta_{1,!}$ is a linear combination of pushforward maps; see [LTX+22, p. 262]. Denote by $B^n(\mathbf{Q}, \mathcal{O}_\lambda)^0$ and $B_n(\mathbf{Q}, \mathcal{O}_\lambda)_0$ the kernel and cokernel of the tautological map

$$B^n(\mathbf{Q}, \mathcal{O}_\lambda) \rightarrow B_{n-1}(\mathbf{Q}, \mathcal{O}_\lambda),$$

respectively. By [Liu19, Lemma 2.4], the composite map

$$H_{\mathbb{F}}^{2(n-1)}(\overline{\mathbf{Q}}^{(0)}, \mathcal{O}_\lambda(n-1)) \xrightarrow{\delta_0^*} H_{\mathbb{F}}^{2(n-1)}(\overline{\mathbf{Q}}^{(1)}, \mathcal{O}_\lambda(n-1)) \xrightarrow{\delta_{1,!}} H_{\mathbb{F}}^{2n}(\overline{\mathbf{Q}}^{(0)}, \mathcal{O}_\lambda(n))$$

factors through a unique map $B_n(\mathbf{Q}, \mathcal{O}_\lambda)_0 \rightarrow B_n(\mathbf{Q}, \mathcal{O}_\lambda)^0$. Set

$$C_n(\mathbf{Q}, \mathcal{O}_\lambda) := B_n(\mathbf{Q}, \mathcal{O}_\lambda)_0^{\mathrm{Gal}_{\mathbb{F}_{p^2}}}, \quad C^n(\mathbf{Q}, \mathcal{O}_\lambda) := B_n(\mathbf{Q}, \mathcal{O}_\lambda)^0_{\mathrm{Gal}_{\mathbb{F}_{p^2}}}.$$

Then we obtain a *potential map*

$$\Delta^n : C_n(\mathbf{Q}, \mathcal{O}_\lambda) \rightarrow C^n(\mathbf{Q}, \mathcal{O}_\lambda).$$

In particular, the cycle $\mathbf{Q}_{\mathrm{sp}}$ gives rise to a class $\mathrm{cl}(\mathbf{Q}_{\mathrm{sp}}) \in C^n(\mathbf{Q}, \mathcal{O}_\lambda)$.

Proposition 3.7.7. *Assume Assumptions 3.7.4, 3.7.5 and Hypothesis 3.2.3 for each $N \in \{n, n+1\}$. There is a canonical isomorphism*

$$H_{\mathrm{sing}}^1 \left(\mathbb{Q}_{p^2}, H_{\mathbb{F}}^{2n-1}(\overline{\mathbf{Q}}, R\Psi\mathcal{O}_\lambda(n))_{(\mathfrak{m}_0, \mathfrak{m}_1)} \right) \cong \mathrm{Coker} \Delta_{(\mathfrak{m}_0, \mathfrak{m}_1)}^n,$$

under which $\partial \mathrm{AJ}_{\mathbf{Q}}(\mathbf{Q}_{\mathrm{sp}}^\eta)$ is identified with the image of $\mathrm{cl}(\mathbf{Q}_{\mathrm{sp}})$ in $\mathrm{Coker} \Delta_{(\mathfrak{m}_0, \mathfrak{m}_1)}^n$.

Proof. Using Lemma 3.7.6, the proof of [LTX+22, Proposition 7.2.7] goes through. $\square$

For each $\alpha \in \{0, 1\}$, we set $\mathrm{Sh}'_{n_\alpha} := \mathrm{Sh}(\mathbf{V}'_{n_\alpha}, j_{n_\alpha} \mathcal{K}_{n_\alpha}^{\circ, p} \mathcal{K}'_{n_\alpha, p})$. By [LTX+22, Construction 5.11.7 and Remark 5.11.8], we obtain a map

$$\nabla : C^n(\mathbf{Q}, \mathcal{O}_\lambda) \rightarrow \mathcal{O}_\lambda[\mathrm{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ)] \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[\mathrm{Sh}(\mathbf{V}_{n_1}^\circ, \mathcal{K}_{n_1}^\circ)].$$

Under Assumption 3.7.3 and Assumption 3.7.5,

$$H_{\mathrm{et}}^{2n}((\mathrm{Sh}'_{n_0} \times_{\mathrm{Spec} F} \mathrm{Sh}'_{n_1})_{\overline{F}}, \mathcal{O}_\lambda)_{(\mathfrak{m}_0, \mathfrak{m}_1)}$$

vanishes. This follows from [LTX+22, Lemma 5.2.7], Lemma 3.4.5, and the Künneth formula. In particular, we obtain an Abel–Jacobi map

$$\mathrm{AJ} : Z^n(\mathrm{Sh}'_{n_0} \times_{\mathrm{Spec} F} \mathrm{Sh}'_{n_1}) \rightarrow H^1 \left(F, H_{\mathrm{et}}^{2n-1}((\mathrm{Sh}'_{n_0} \times_{\mathrm{Spec} F} \mathrm{Sh}'_{n_1})_{\overline{F}}, \mathcal{O}_\lambda(n))_{(\mathfrak{m}_0, \mathfrak{m}_1)} \right)$$

and its natural projection

$$\overline{\text{AJ}} : Z^n(\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1}) \rightarrow H^1(F, H_{\text{et}}^{2n-1}((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\overline{F}}, \mathcal{O}_\lambda(n)) / (\mathbf{n}_0, \mathbf{n}_1)).$$

Let Sh'_{sp} denote the cycle given by the finite morphism $\text{Sh}(\mathbf{V}'_n, \mathbf{j}_n \mathcal{K}_{\text{sp}}^{\circ, p} \mathcal{K}'_{n, p}) \rightarrow \text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1}$.

Proposition 3.7.8. *Assume Assumptions 3.7.4, 3.7.5 and Hypothesis 3.2.3 for each $N \in \{n, n+1\}$.*

(1) *The map ∇ descends modulo $(\mathbf{n}_0, \mathbf{n}_1)$ to an isomorphism*

$$\nabla_{/(\mathbf{n}_0, \mathbf{n}_1)} : \text{Coker } \Delta^n / (\mathbf{n}_0, \mathbf{n}_1) \xrightarrow{\sim} \mathcal{O}_\lambda[\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ)] \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathcal{K}_{n_1}^\circ)] / (\mathbf{n}_0, \mathbf{n}_1).$$

(2) *The Hecke operator $(p+1)\mathbf{I}_{n_0, \mathbf{p}}^\circ \otimes \mathbf{T}_{n_1, \mathbf{p}}^\circ$ acts invertible on*

$$\mathcal{O}_\lambda[\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ)] \otimes_{\mathcal{O}_\lambda} \mathcal{O}_\lambda[\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathcal{K}_{n_1}^\circ)] / (\mathbf{n}_0, \mathbf{n}_1);$$

denote its inverse by $\mathbf{T}^\circ$. Moreover,

$$\nabla_{/(\mathbf{n}_0, \mathbf{n}_1)}(\partial_{\mathbf{p}} \text{AJ}_{\mathbf{Q}})(\mathbf{Q}_{\text{sp}}^\eta) = \mathbf{T}^\circ \mathbf{1}_{\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)},$$

where $\mathbf{1}_{\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)}$ is the pushforward of the characteristic function along the map $\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ) \rightarrow \text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_n^\circ) \times \text{Sh}(\mathbf{V}_{n+1}^\circ, \mathcal{K}_{n+1}^\circ)$.

(3)

$$\begin{aligned} & \exp_\lambda(\partial_{\mathbf{p}} \text{loc}_{\mathbf{p}} \overline{\text{AJ}}(\text{Sh}'_{\text{sp}}), H_{\text{sing}}^1(F_{\mathbf{p}}, H_{\text{et}}^{2n-1}((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\overline{F}}, \mathcal{O}_\lambda(n)) / (\mathbf{n}_0, \mathbf{n}_1))) \\ &= \exp_\lambda(\mathbf{1}_{\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)}, \mathcal{O}_\lambda[\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ) \times \text{Sh}(\mathbf{V}_{n_1}^\circ, \mathcal{K}_{n_1}^\circ)] / (\mathbf{n}_0, \mathbf{n}_1)). \end{aligned}$$

Proof. For (1): We follow the proof of [LTX⁺22, Theorem 7.2.8(2)]. Firstly, by Proposition 3.7.6(1), Proposition 3.5.2(4) and Proposition 3.6.3(3), the map $\nabla_{/(\mathbf{n}_0, \mathbf{n}_1)}$ is surjective. Thus it remains to show that the domain and the target of $\nabla_{/(\mathbf{n}_0, \mathbf{n}_1)}$ are isomorphic as $\mathcal{O}_\lambda$ -modules. By the proof of [LTX⁺22, Theorem 7.2.8(2)], this follows from Proposition 3.7.7, Lemma 3.7.6(2, 3), Proposition 3.5.2(4), and Proposition 3.6.3(4, 5).

For (2): $p+1$ is invertible in $\mathcal{O}_\lambda$ by (PI2); $\mathbf{I}_{n_0, \mathbf{p}}^\circ \otimes \mathbf{T}_{n_1, \mathbf{p}}^\circ$ is invertible by (PI4, PI5), [LTX⁺22, Propositions B.3.5(1), B.4.3(2)] and [LTX24, Lemma 4.2.4(1)];

For (3): This follows from part (2) by the proof of [LTX⁺22, Corollary 7.2.9]. $\square$

3.8. Admissible places. We now work in the setting of Setup 3.7.1.

Definition 3.8.1. We say that a finite place $\lambda \in \Sigma_E^{\text{fin}}$, with underlying prime ℓ , is an admissible place (with respect to (Π_0, Π_1)) if the following hold:⁷

- (L1) $\ell \geq 2(n_0 + 1)$;
- (L2) $\Sigma_+^{\Pi_0}$ does not contain places lying above ℓ ;
- (L3) The residual representations $\overline{\rho}_{\Pi_0, \lambda}$ and $\overline{\rho}_{\Pi_1, \lambda}$ are both absolutely irreducible. Fix Gal_F -stable $\mathcal{O}_\lambda$ -lattices $R_0 \subset \rho_{\Pi_0, \lambda}(r_0)$ and $R_1^b \subset \rho_{\Pi_1, \lambda}(r_1)$ (which are unique up to homothety), together with isomorphisms $\Xi_0 : R_0 \xrightarrow{\sim} R_0^\vee(1)$ and $\Xi_1^b : R_1^b \xrightarrow{\sim} (R_1^b)^\vee$. Set $R_1 := R_1^b \oplus \mathcal{O}_\lambda$ and $\Xi_1 := \Xi_1^b \oplus \text{id} : R_1 \xrightarrow{\sim} R_1^\vee$.
- (L4-1) One of the following holds:
 - (a) The image of Gal_F in $\text{GL}(\overline{R_0})$ contains a nontrivial scalar element;
 - (b) $\overline{R_0}$ is a semisimple $\kappa_\lambda[\text{Gal}_F]$ -module and $\text{Hom}_{\kappa_\lambda[\text{Gal}_F]}(\text{End}(\overline{R_0}), \overline{R_0}) = 0$;
- (L4-2) $(\text{GI}_{F', \mathcal{P}, R_0, R_1}^1)$ from Lemma 2.3.3 holds for $F' = F_{\text{rfx}, +}$ and $\mathcal{P}(T) = T^2 - 1$;
- (L5) The homomorphism $\overline{\rho}_{\Pi_0, \lambda, +}$ is rigid for $(\Sigma_+^{\Pi_0}, \emptyset)$ (see Definition 3.6.1), and $\overline{\rho}_{\Pi_0, \lambda}|_{\text{Gal}_F(\mu_\ell)}$ is absolutely irreducible; and
- (L6) The composite homomorphism $\mathbb{T}_{n_\alpha}^{\Sigma^{\Pi_0} \cup \Sigma^{\Pi_1}} \xrightarrow{\phi_{\Pi_\alpha}} \mathcal{O}_E \rightarrow \kappa_\lambda$ is cohomologically generic (Definition 3.2.5) for every $\alpha \in \{0, 1\}$.

To end this subsection, we give several examples where it is known that all but finitely many finite places λ of E are admissible.

⁷Compared to [LTX⁺22, Definition 8.1.1], we omitted assumption (L3) because we will not consider the Bloch–Kato Selmer group of the Galois representation $\rho_{\Pi_0, \lambda} \otimes \rho_{\Pi_1, \lambda}$.

Lemma 3.8.2. *Suppose that*

- (1) *there exists an elliptic curve A_0 over F_+ such that for every finite place λ of E ,*

$$\rho_{\Pi_0, \lambda} \cong \text{Sym}^{n_0-1} H_{\text{ét}}^1(A_{\overline{F}}, E_\lambda)|_{\text{Gal}_F};$$

- (2) *there exists a good inert place $\mathfrak{p}$ of F_+ (see Definition 3.3.3) such that A_0 has split multiplicative reduction at $\mathfrak{p}$, and $\Pi_{1, \mathfrak{p}}^\mathfrak{p}$ is a supercuspidal B -avoiding good representation (see Definition A.1.2) for*

$$B = \{-\|\mathfrak{p}\|, \|\mathfrak{p}\|^{1\pm 1}, \|\mathfrak{p}\|^{1\pm 3}, \dots, \|\mathfrak{p}\|^{1\pm(2r-1)}\}$$

with respect to any isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ where ℓ is not a rational prime underlying $\mathfrak{p}$. Then all but finitely many finite places λ of E are admissible (with respect to (Π_0, Π_1)).

Proof. We show that every condition in Definition 3.8.1 excludes only finitely many finite places of E . By [Ser72, Théorème 6], for sufficiently large prime ℓ , the homomorphism

$$\bar{\rho}_{A, \ell}|_{\text{Gal}_F} : \text{Gal}_F \rightarrow \text{GL}(H_{\text{ét}}^1(A_{\overline{F}}, \mathbb{F}_\ell))$$

is surjective. So we may assume that ℓ is large such that this is the case.

For (L1) and (L2), this is trivial.

For (L3), $\bar{\rho}_{\Pi_0, \lambda}$ is clearly absolutely irreducible, and the condition that $\bar{\rho}_{\Pi_1^\mathfrak{p}, \lambda}$ is absolutely irreducible only excludes finitely many finite places λ of E by [LTX⁺24, Theorem 4.5.(1)] and condition (2).

For (L4-1), condition (a) always holds.

For (L4-2), because A_0 has split multiplicative reduction at $\mathfrak{p}$, $\Pi_{0, \mathfrak{p}}$ is the Steinberg representation by [Roh94, §15]. Thus (L4-2) excludes only finitely many finite places λ of E , by the same reasoning as in the proof of [LTX⁺22, Lemma 8.1.4].

For (L5), by [LTX⁺24, Corollary 4.2], the condition that $\bar{\rho}_{\Pi_0, \lambda, +}$ is rigid for $(\Sigma_+^{\min}, \emptyset)$ excludes only finitely many finite places λ of E . The second condition is clearly satisfied.

For (L6), for each $\alpha \in \{0, 1\}$, we choose a finite place w_α of F such that Π_{α, w_α} is unramified with Satake parameter $\{a_{\alpha, 1}, \dots, a_{\alpha, n_\alpha}\}$. By Proposition 2.1.1, $|a_{\alpha, i}| = 1$ for every $1 \leq i \leq n_\alpha$. Thus, for every sufficiently large rational prime ℓ , $a_{\alpha, i}/a_{\alpha, j} \neq \|w\|$ for $1 \leq i \neq j \leq n_\alpha$ even in $\overline{\mathbb{F}_\ell}$. Suppose λ is a finite place of E lying above ℓ . We fix an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ which induces λ . Applying the Chebotarev density theorem to the representation $\bar{\rho}_{\Pi, \lambda} \oplus \bar{\varepsilon}_\ell$ of Gal_F , we see that there are infinitely many finite places w'_α of F that are of degree 1 over $\mathbb{Q}$ satisfying that

- Π_{α, w'_α} is unramified with Satake parameter $\{a'_{\alpha, 1}, \dots, a'_{\alpha, n_\alpha}\}$ in which $\iota_\ell(a'_{\alpha, i})$ is an ℓ -adic unit for every $1 \leq i \leq n_\alpha$, and
- $\iota_\ell(a'_{\alpha, i}/a'_{\alpha, j}) \neq \|w'_\alpha\| \in \overline{\mathbb{F}_\ell}$ for $1 \leq i \neq j \leq n_\alpha$.

Then it follows from [YZ25, Theorem 1.5] that (L6) holds for λ . $\square$

Lemma 3.8.3. *Suppose that*

- (1) *there exists a very good inert place $\mathfrak{p}$ of F_+ (see Definition 3.3.3) such that $\Pi_{0, \mathfrak{p}}$ is Steinberg, and $\Pi_{1, \mathfrak{p}}^\mathfrak{p}$ is unramified with Satake parameter not containing 1; and*

- (2) *for each $\alpha \in \{0, 1\}$, there exist a finite place w_α of F such that Π_{α, w_α} is supercuspidal;*

Then all but finitely many finite places λ of E are admissible (with respect to (Π_0, Π_1)).

Proof. We show that every condition in Definition 3.8.1 excludes only finitely many finite places of E .

For (L1) and (L2), this is trivial.

For (L3), this follows from [LTX⁺24, Theorem 4.5.(1)] by (2).

For (L5), this follows from [LTX⁺24, Theorem 4.8] by (2).

For (L6), this follows from the same reasoning as in the proof of Lemma 3.8.2.

For (L4-1), this follows by the same reasoning as in the proof of [LTX⁺22, Lemma 8.1.4].

For (L4-2), this follows by the same reasoning as in the proof of [LTX⁺22, Lemma 8.1.4]. $\square$

3.9. Proof of Theorem D. The following lemma is crucial for the proof of Theorem D, which is essentially the solution of the Gan–Gross–Prasad conjecture for unitary groups [JR11, Zha14, BPLZZ21, BPCZ22].

Lemma 3.9.1. *We work in the setting of Setup 3.7.1. If $L(\frac{1}{2}, \Pi_0 \times \Pi_1) \neq 0$, then there exists*

- a standard definite Hermitian space $\mathbf{V}_n^\circ$ of dimension n over F , together with a self-dual $\prod_{v \in \Sigma_+^{\text{fin}} \setminus (\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1})} \mathcal{O}_{F_v}$ -lattice Λ_n° in $\mathbf{V}_n^\circ \otimes_{F_+} \mathbf{A}_{F_+}^{\Sigma_+^\infty \cup \Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}}$, and we set $\mathbf{V}_{n+1}^\circ = (\mathbf{V}_n^\circ)_\#$ and $\Lambda_{n+1}^\circ = (\Lambda_n^\circ)_\#$.
- objects $\mathcal{K}_n \in \mathfrak{K}(\mathbf{V}_n^\circ)$ and $(\mathcal{K}_{\text{sp}}^\circ, \mathcal{K}_{n+1}^\circ) \in \mathfrak{K}(\mathbf{V}_n^\circ)_{\text{sp}}$ of the forms

$$\begin{aligned}\mathcal{K}_n^\circ &= \prod_{v \in \Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}} (\mathcal{K}_n^\circ)_v \times \prod_{v \in \Sigma_+^{\text{fin}} \setminus (\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1})} \text{U}(\Lambda_n^\circ)(\mathcal{O}_v), \\ \mathcal{K}_{\text{sp}}^\circ &= \prod_{v \in \Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}} (\mathcal{K}_{\text{sp}}^\circ)_v \times \prod_{v \in \Sigma_+^{\text{fin}} \setminus (\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1})} \text{U}(\Lambda_n^\circ)(\mathcal{O}_v), \\ \mathcal{K}_{n+1}^\circ &= \prod_{v \in \Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}} (\mathcal{K}_{n+1}^\circ)_v \times \prod_{v \in \Sigma_+^{\text{fin}} \setminus (\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1})} \text{U}(\Lambda_{n+1}^\circ)(\mathcal{O}_v),\end{aligned}$$

satisfying

- $\mathcal{K}_{\text{sp},v}^\circ \subset \mathcal{K}_{n,v}^\circ$ for $v \in \Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}$, and
- $\mathcal{K}_{n_0,v}^\circ$ is hyperspecial maximal subgroup of $\text{U}(\mathbf{V}_{n_\alpha}^\circ)(F_v)$ for $v \in \Sigma_+^{\Pi_0} \setminus \Sigma_+^{\Pi_1}$,

such that

$$\sum_{s \in \text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)} f(s) \neq 0$$

for some $f \in \mathcal{O}_E [\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ)] [\ker \phi_{\Pi_0}] \otimes_{\mathcal{O}_E} \mathcal{O}_E [\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathcal{K}_{n_1}^\circ)] [\ker \phi_{\Pi_1}]$. Here we regard f as a function on $\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)$ via the map $\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ) \rightarrow \text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_n^\circ) \times \text{Sh}(\mathbf{V}_{n+1}^\circ, \mathcal{K}_{n+1}^\circ)$.

Proof. In view of Remark 1.1.4, this follows from the direction (1) $\implies$ (2) of [BPCZ22, Theorem 1.1.5.1]. Note that since our Π_n and Π_{n+1} are relevant representations of $\text{GL}_n(\mathbf{A}_F)$ and $\text{GL}_{n+1}(\mathbf{A}_F)$, respectively, the Hermitian space in (2) of [BPCZ22, Theorem 1.1.5.1] is standard definite. $\square$

Theorem 3.9.2. *We work in the setting of Setup 3.7.1. Assume there is a finite place w of F lying above a place of F_+ inert in F such that $(\Pi_1^\flat)_w$ is square-integrable, and assume Hypothesis 3.2.3 for each $N \in \{n, n+1\}$. If the central critical value*

$$L(\tfrac{1}{2}, \Pi_0) \cdot L(\tfrac{1}{2}, \Pi_0 \times \Pi_1^\flat)$$

does not vanish, then for all admissible finite places λ of E (with respect to (Π_0, Π_1)), the Bloch–Kato Selmer group $H_f^1(F, \rho_{\Pi_0, \lambda}(r_0))$ vanishes.

Proof. The proof is a variant of that of [LTX⁺22, Theorem 8.2.2]. By Lemma 3.9.1, we may fix the choices of $\mathbf{V}_n^\circ, \mathbf{V}_{n+1}^\circ, \Lambda_n^\circ, \Lambda_{n+1}^\circ; \mathcal{K}_n^\circ, \mathcal{K}_{\text{sp}}^\circ, \mathcal{K}_{n+1}^\circ$ in that lemma such that

$$\sum_{s \in \text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)} f(s) \neq 0$$

for some $f \in \mathcal{O}_E [\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ)] [\ker \phi_{\Pi_0}] \otimes_{\mathcal{O}_E} \mathcal{O}_E [\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathcal{K}_{n_1}^\circ)] [\ker \phi_{\Pi_1}]$.

Let λ be an admissible finite place of E with the underlying rational prime ℓ . We choose a Gal_F -stable $\mathcal{O}_\lambda$ -lattice $\mathbf{R}_0$ in $\rho_{\Pi_0, \lambda}(r_\alpha)$, unique up to homothety, with a fixed isomorphism $\Xi_0 : \mathbf{R}_0 \xrightarrow{\sim} \mathbf{R}_0^\vee(1)$; and a Gal_F -stable $\mathcal{O}_\lambda$ -lattice $\mathbf{R}_1^\flat$ in $\rho_{\Pi_1^\flat, \lambda}(r_1)$, unique up to homothety, with a fixed isomorphism $\Xi_1^\flat : \mathbf{R}_1^\flat \xrightarrow{\sim} (\mathbf{R}_1^\flat)^\vee$. Set $\mathbf{R}_1 := \mathbf{R}_1^\flat \oplus \mathcal{O}_\lambda$, with a fixed isomorphism $\Xi : \mathbf{R}_1 \xrightarrow{\sim} \mathbf{R}_1^\vee$. We write $\mathbf{R} := \mathbf{R}_0 \otimes \mathbf{R}_1$ and $\Xi := \Xi_0 \otimes \Xi_1 : \mathbf{R} \xrightarrow{\sim} \mathbf{R}^\vee(1)$. Define two nonnegative integers m_{per} and m_{lat} as follows.

- (1) Let m_{per} denote the largest nonnegative integer such that

$$\sum_{s \in \text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)} f(s) \in \lambda^{m_{\text{per}}} \mathcal{O}_E$$

for every $f \in \mathcal{O}_E [\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ)] [\ker \phi_{\Pi_0}] \otimes_{\mathcal{O}_E} \mathcal{O}_E [\text{Sh}(\mathbf{V}_{n_1}^\circ, \mathcal{K}_{n_1}^\circ)] [\ker \phi_{\Pi_1}]$.

- (2) We choose a standard indefinite Hermitian space $\mathbf{V}_{n_1}$ over F of rank n_1 , together with a fixed isomorphism $U((\mathbf{V}_{n_1}^\circ)^\infty) \cong U(\mathbf{V}_{n_1}^\infty)$ of reductive groups over $\mathbf{A}_{F_+}^\infty$. In particular, we obtain the Shimura variety $\text{Sh}(\mathbf{V}_{n_1}, \mathcal{K}_{n_1}^\circ)$. By Hypothesis 3.2.3, there is an isomorphism

$$H_{\text{ét}}^{2r_1}(\text{Sh}(\mathbf{V}_{n_1}, \mathcal{K}_{n_1}^\circ)_{\overline{F}}, E_\lambda(r_1)) / \ker \phi_{\Pi_1} \cong (R_1^c \otimes_{\mathcal{O}_\lambda} E_\lambda)^{\oplus \mu_1}$$

of $E_\lambda[\text{Gal}_F]$ -modules for some positive integer $\mu_1 \in \mathbb{Z}_+$. We fix a map

$$H_{\text{ét}}^{2r_1}(\text{Sh}(\mathbf{V}_{n_1}, \mathcal{K}_{n_1}^\circ)_{\overline{F}}, \mathcal{O}_\lambda(r_1)) / \ker \phi_{\Pi_1} \rightarrow (R_1^c)^{\oplus \mu_1}$$

of $\mathcal{O}_\lambda[\text{Gal}_F]$ -modules whose kernel and cokernel are both $\mathcal{O}_\lambda$ -torsion. Then we denote by m_{lat} the smallest nonnegative integer such that both the kernel and the cokernel are annihilated by $\lambda^{m_{\text{lat}}}$.

We start to prove the theorem by contradiction, hence assume

$$\dim_{E_\lambda} H_f^1(F, \rho_{\Pi_0, \lambda}(r_0)) \geq 1.$$

Tate a sufficiently large positive integer m which will be determined later. By Lemma 2.1.3, we may apply [LTX⁺22, Proposition 2.4.6] by taking Σ to be the set of places of F lying above $\Sigma_+^{\Pi_0}$. Then we obtain a submodule S of $H_{f, R}^1(F, \overline{R_0}^{(m)})$ that is free of rank 1 over $\mathcal{O}_\lambda / \lambda^{m-m_\Sigma}$ such that $\text{loc}_w|_S = 0$ for every finite place w of F lying above $\Sigma_+^{\Pi_0}$. We now apply the discussion of [LTX⁺22, §2.3] to the submodule $S \subset H^1(F, \overline{R_0}^{(m)})$. By (L4-1) and [LTX⁺22, Lemma 2.3.4], we obtain an injective map

$$\theta_S : \text{Gal}(F_S / F_{\overline{\rho}^{(m)}}) \rightarrow \text{Hom}_{\mathcal{O}_\lambda}(S, \overline{R_0}^{(m)})$$

whose image generates an $\mathcal{O}_\lambda$ -submodule containing $\lambda^{\mathfrak{r}_{R_0}^{(m)}} \text{Hom}_{\mathcal{O}_\lambda}(S, \overline{R_0}^{(m)})$, which further contains $\lambda^{\mathfrak{r}_{R_0}} \text{Hom}_{\mathcal{O}_\lambda}(S, \overline{R_0}^{(m)})$ by [LTX⁺22, Lemma 2.3.3] and (L3) (Here $\mathfrak{r}_{\overline{R_0}^{(m)}}$ and $\mathfrak{r}_{R_0}$ are reducibility depths defined in [LTX⁺22, Definition 2.3.2, Proposition 2.3.3]). By (L4-2) and Lemma 2.3.3, we may choose an element $(\gamma_0, \gamma_1, \xi)$ in the image of $(\overline{\rho}_{\Pi_0, \lambda}^{(m)}, \overline{\rho}_{\Pi_1, \lambda}^{(m)}, \overline{\varepsilon}_\ell^{(m)})|_{\text{Gal}_{F_{\text{fix}, +}}}$ satisfying conditions (a-d) in Lemma 2.3.3. In particular, the natural inclusion

$$(3.6) \quad (\overline{R_0}^{(m)})^{h_{\gamma_0}} \rightarrow (\overline{R}^{(m)})^{h_{\gamma_0} \otimes h_{\gamma_1}}$$

is an isomorphism of free $\mathcal{O}_\lambda / \lambda^m$ -modules of rank 1. By [LTX⁺22, Proposition 2.6.6] (with $m_0 = m_\Sigma$ and $r_S = 1$), we may fix an (S, γ) -abundant element $\Psi \in G_{S, \gamma}$ (see [LTX⁺22, Definition 2.6.5]).

By the Chebotarev density theorem, we can choose a γ -associated place (see [LTX⁺22, Definition 2.6.3]) $w_+^{(m)}$ of $F_+^{(m)}$ satisfying $\Psi_{w_+^{(m)}} = \Psi$ and whose underlying prime $\mathfrak{p}$ of F_+ (with its underlying rational prime p and an isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$ under which τ_∞ and $\mathfrak{p}$ correspond) is a very good inert place satisfying (PI1)-(PI5) and

(PI6) the natural map

$$\frac{H_{\text{ét}}^{2r_1}(\text{Sh}(\mathbf{V}_{n_1}, \mathcal{K}_{n_1}^\circ)_{\overline{F}}, \mathcal{O}_\lambda(r_1))}{\mathbb{T}_{n_1}^{\Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1} \cup \Sigma_{F_+}(p)} \cap \ker \phi_{\Pi_1}} \rightarrow \frac{H_{\text{ét}}^{2r_1}(\text{Sh}(\mathbf{V}_{n_1}, \mathcal{K}_{n_1}^\circ)_{\overline{F}}, \mathcal{O}_\lambda(r_1))}{\ker \phi_{\Pi_1}}$$

is an isomorphism.

We can choose a quintuple $\mathcal{T} = (\Phi, \mathbf{W}_0, \mathcal{K}_0^p, \iota_p, \varpi)$ as in [LTX⁺22, §5.1] with $\mathbb{Q}_p^\Phi = \mathbb{Q}_{p^2}$, an octuple

$$\mathcal{V}^\bullet = (\Lambda_{n, \mathfrak{p}}^\bullet, \Lambda_{n+1, \mathfrak{p}}^\bullet; \mathcal{K}_{n, p}^\bullet, \mathcal{K}_{n+1, p}^\bullet, \mathcal{K}_{\text{sp}, p}^\bullet; \mathcal{K}_{n, p}^\dagger, \mathcal{K}_{\text{sp}, p}^\dagger, \mathcal{K}_{n+1, p}^\dagger)$$

as in [LTX⁺22, Notation 5.10.13], and a sextuple $\mathcal{U}$ as in Setup 3.7.2. We are now working in the setting of Setup 3.7.2 with

$\lambda, \quad \Sigma_+^{\text{lr}, \text{I}} = \emptyset, \quad \Sigma_+^{\text{I}} = \Sigma_+^{\Pi_0} \cup \Sigma_+^{\Pi_1}, \quad \mathcal{V}^\circ = (\mathbf{V}_n^\circ, \mathbf{V}_{n+1}^\circ; \Lambda_n^\circ, \Lambda_{n+1}^\circ; \mathcal{K}_n^\circ, \mathcal{K}_{\text{sp}}^\circ, \mathcal{K}_{n+1}^\circ), \quad m, \quad \mathfrak{p}, \quad \mathcal{T}, \quad \mathcal{V}^\bullet, \quad \mathcal{U}$
specified.

By the definition of m_{per} ,

$$(3.7) \quad \exp_\lambda \left(\mathbf{1}_{\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)}, \mathcal{O}_\lambda [\text{Sh}(\mathbf{V}_{n_0}^\circ, \mathcal{K}_{n_0}^\circ) \times \text{Sh}(\mathbf{V}_{n_1}^\circ, \mathcal{K}_{n_1}^\circ)] / (\mathbf{n}_0, \mathbf{n}_1) \right) \geq m - m_{\text{per}},$$

where $\mathbf{1}_{\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ)}$ is the pushforward of the characteristic function along the map $\text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_{\text{sp}}^\circ) \rightarrow \text{Sh}(\mathbf{V}_n^\circ, \mathcal{K}_n^\circ) \times \text{Sh}(\mathbf{V}_{n+1}^\circ, \mathcal{K}_{n+1}^\circ)$.

We claim that there exists an element $c_1 \in H^1(F, \overline{R_0}^{(m),c})$ such that

$$(3.8) \quad \exp_\lambda \left(\partial_{\mathfrak{p}} \text{loc}_{\mathfrak{p}}(c_1), H_{\text{sing}}^1(F_{\mathfrak{p}}, \overline{R_0}^{(m),c}) \right) \geq m - m_{\text{per}} - m_{\text{lat}},$$

and for every finite place w of F not lying above $\Sigma_+^{\Pi_0} \cup \{\mathfrak{p}\}$,

$$(3.9) \quad \text{loc}_w(c_1) \in H_{\text{ns}}^1(F_w, \overline{R_0}^{(m),c}).$$

We first prove the theorem assuming the existence of such c_1 . Fix a generator s_1 of the submodule $S \subset H_{f,R_0}^1(F, \overline{R}^{(m)})$. We also identify $\overline{R_0}^{(m),c}$ with $(\overline{R_0}^{(m)})^*(1)$ via the polarization Ξ_0 . We now compute the local Tate pairing $\langle s_1, c_1 \rangle_w$ (see [LTX+22, Equation (2.2)]) for every finite place w of F .

- Suppose w is lying above $\Sigma_+^{\Pi_0}$. Then $\text{loc}_w(s_1)$ vanishes by our choice of S . Thus $\langle s_1, c_1 \rangle_w = 0$.
- Suppose w is lying above $\Sigma_{F_+}(\ell)$. Then by (L2), $(R_0)_{\mathbb{Q}}$ is crystalline with Hodge–Tate weights in $[-r_0, r_0 + 1]$. Thus $\text{loc}_w(c_1)$ is in $H_{\text{ns}}^1(F_w, \overline{R}^{(m)})$ by [LTX+22, Lemma 2.4.3(2)] and (L1). By (3.9), [LTX+22, Lemma 2.2.7] and (L1), $\lambda^{m_{\text{dif}}} \langle s_1, c_1 \rangle_w$ vanishes, where $\mathfrak{d}_\lambda = \lambda^{m_{\text{dif}}} \subset \mathcal{O}_\lambda$ is the different ideal of E_λ over $\mathbb{Q}_\ell$.
- Suppose w is not lying above $\Sigma_+^{\Pi_0} \cup \Sigma_{F_+}(\ell) \cup \{\mathfrak{p}\}$. Then by (L2), R_0 is unramified. Thus $\text{loc}_w(c_1)$ is in $H_{\text{ns}}^1(F_w, \overline{R}^{(m)})$ by [LTX+22, Lemma 2.4.3(1)]. By (3.9) and [LTX+22, Lemma 2.2.3], $\langle s_1, c_1 \rangle_w$ vanishes.
- Suppose w is the unique place lying above $\mathfrak{p}$. Then

$$\exp_\lambda \left(\text{loc}_w(s_1), H_{\text{ns}}^1(F_w, \overline{R_0}^{(m)}) \right) \geq m - m_\Sigma - \mathfrak{r}_{R_0}$$

by [LTX+22, Proposition 2.6.7]; and

$$\exp_\lambda (\langle s_1, c_1 \rangle_w, \mathcal{O}_\lambda / \lambda^m) \geq m - m_{\text{per}} - m_{\text{lat}} - m_\Sigma - \mathfrak{r}_{R_0}$$

by (3.8) and [Rub00, Proposition I.4.3.(ii)].

Therefore, as long as we take m such that $m > m_{\text{per}} + m_{\text{lat}} + m_\Sigma + \mathfrak{r}_{R_0} + m_{\text{dif}}$, we will have a contradiction to the relation

$$\sum_{w \in \Sigma_F^\infty} \langle s_1, c_1 \rangle_w = 0.$$

The theorem is proved assuming the claim.

We now consider the claim on the existence of c_1 . It follows from (L5), (L6) and Proposition 3.6.3(6) that there exists an isomorphism

$$\overline{\Upsilon}_0 : H_{\text{ét}}^{2r_0-1} \left(\text{Sh}(\mathbf{V}'_{n_0}, \mathbf{j}_{n_0} \mathcal{K}_{n_0}^{\infty,p} \mathcal{K}'_{n_0,p})_{\overline{F}}, \mathcal{O}_\lambda(r_0) \right) / \mathfrak{n}_0 \xrightarrow{\sim} \left(\overline{R_0}^{(m),c} \right)^{\oplus \mu_0}$$

of $\mathcal{O}_\lambda[\text{Gal}_F]$ -modules, for some positive integer $\mu_0 \in \mathbb{Z}_+$. It follows from Lemma 3.4.7 that there exists an isomorphism

$$H_{\text{ét}}^{2r_1}(\text{Sh}(\mathbf{V}_{n_1}, \mathcal{K}_{n_1}^{\circ})_{\overline{F}}, \mathcal{O}_\lambda)_{\mathfrak{m}_1} \cong H_{\text{ét}}^{2r_1}(\text{Sh}(\mathbf{V}'_{n_1}, \mathbf{j}_{n_1}(\mathcal{K}_{n_1}^{p,\circ}) \mathcal{K}'_{n_1,p})_{\overline{F}}, \mathcal{O}_\lambda)_{\mathfrak{m}_1}$$

of $\mathcal{O}_\lambda[\text{Gal}_F]$ -modules. Thus, by (PI8) and the definition of m_{lat} , we may fix a map

$$\Upsilon_1 : \frac{H_{\text{ét}}^{2r_1}(\text{Sh}(\mathbf{V}'_{n_1}, \mathbf{j}_{n_1}(\mathcal{K}_{n_1}^{p,\circ}) \mathcal{K}'_{n_1,p})_{\overline{F}}, \mathcal{O}_\lambda(r_1))}{\mathbb{T}_{n_1}^{\Sigma_{n_0}^{\Pi_0} \cup \Sigma_+^{\Pi_1} \cup \Sigma_{F_+}(p)} \cap \ker \phi_{\Pi_1}} \rightarrow (R_1^{\mathfrak{c}})^{\oplus \mu_1}$$

of $\mathcal{O}_\lambda[\text{Gal}_F]$ -modules whose kernel and cokernel are both annihilated by $\lambda^{m_{\text{lat}}}$.

To continue, we adopt the notational abbreviation prior to Proposition 3.7.8. By Lemma 3.4.5 and the Künneth formula, we obtain a map

$$\overline{\Upsilon} := \overline{\Upsilon}_0 \otimes \Upsilon_1 : H_{\text{ét}}^{2n-1} \left((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\overline{F}}, \mathcal{O}_\lambda(n) \right) / (\mathfrak{n}_0, \mathfrak{n}_1) \rightarrow \left(\overline{R}^{(m),c} \right)^{\oplus \mu_0 \mu_1}$$

of $\mathcal{O}_\lambda[\text{Gal}_F]$ -modules whose kernel and cokernel are both annihilated by $\lambda^{m_{\text{lat}}}$. Consider the class

$$\overline{\text{AJ}}(\text{Sh}'_{\text{sp}}) \in H^1(F, H_{\text{ét}}^{2n-1} \left((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\overline{F}}, \mathcal{O}_\lambda(n) \right) / (\mathfrak{n}_0, \mathfrak{n}_1)).$$

Here Sh'_{sp} denotes the cycle associated to the finite morphism $\text{Sh}(\mathbf{V}'_n, j_n \mathcal{K}_{\text{sp}}^{\circ, p} \mathcal{K}'_{n, p}) \rightarrow \text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1}$. It follows from Proposition 3.7.8(3) and (3.7) that

$$(3.10) \quad \exp_{\lambda} \left(\partial_{\mathbf{p}} \text{loc}_{\mathbf{p}} \overline{\text{AJ}}(\text{Sh}'_{\text{sp}}), H_{\text{sing}}^1(F_{\mathbf{p}}, H_{\text{ét}}^{2n-1}((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\overline{F}}, \mathcal{O}_{\lambda}(n)) / (\mathbf{n}_0, \mathbf{n}_1)) \right) \geq m - m_{\text{per}}.$$

For each $1 \leq i \leq \mu_0$ and each $1 \leq j \leq \mu_1$, let

$$\begin{aligned} \overline{\Upsilon}_{i,j} : H_{\text{ét}}^{2n-1}((\text{Sh}'_{n_0} \times_{\text{Spec } F} \text{Sh}'_{n_1})_{\overline{F}}, \mathcal{O}_{\lambda}(n)) / (\mathbf{n}_0, \mathbf{n}_1) &\xrightarrow{\overline{\Upsilon}} \left(\overline{\mathbf{R}}^{(m), \mathbf{c}} \right)^{\oplus \mu_0 \mu_1} \\ &= \left(\overline{\mathbf{R}}_0^{(m), \mathbf{c}} \right)^{\oplus \mu_0 \mu_1} \oplus \left(\overline{\mathbf{R}}_0 \otimes \mathbf{R}_1^{\flat} \right)^{(m), \mathbf{c}}{}^{\oplus \mu_0 \mu_1} \\ &\xrightarrow{\text{pr}_{i,j}} \overline{\mathbf{R}}_0^{(m), \mathbf{c}} \end{aligned}$$

denote the composition of $\overline{\Upsilon}$ with the projection to the (i, j) -th $\overline{\mathbf{R}}_0^{(m), \mathbf{c}}$ -factor, and set

$$c_{i,j} := H^1(F, \overline{\Upsilon}_{i,j})(\text{AJ}(\text{Sh}'_{\text{sp}})) \in H^1(F, \overline{\mathbf{R}}_0^{(m), \mathbf{c}}).$$

Then it follows from (3.10) and (3.6) that

$$\max_{1 \leq i \leq \mu_0} \max_{1 \leq j \leq \mu_1} \exp_{\lambda} \left(\partial_{\mathbf{p}} \text{loc}_{\mathbf{p}}(c_{i,j}), H_{\text{sing}}^1(F_{\mathbf{p}}, \overline{\mathbf{R}}_0^{(m), \mathbf{c}}) \right) \geq m - m_{\text{per}} - m_{\text{lat}}.$$

Thus we obtain (3.8) by taking $c_1 = c_{i,j}$ for some i, j . On the other hand, by (L6),

$$\mathcal{H}_{\alpha} := H_{\text{ét}}^{n_{\alpha}-1}((\text{Sh}'_{n_{\alpha}})_{\overline{F}}, \mathcal{O}_{\lambda}(n))_{\mathfrak{m}_{\alpha}}$$

is a finite free $\mathcal{O}_{\lambda}$ -module for each $\alpha \in \{0, 1\}$. By Lemma 3.4.5 and the Künneth formula, the following composition map

$$\mathcal{H}_0 \otimes_{\mathcal{O}_{\lambda}} \mathcal{H}_1 \xrightarrow{1 \otimes \Upsilon_1} \mathcal{H}_0 \otimes_{\mathcal{O}_{\lambda}} (\mathbf{R}_1^{\flat, \mathbf{c}} \oplus \mathcal{O}_{\lambda})^{\oplus \mu_1} \xrightarrow{1 \otimes \text{pr}_j} \mathcal{H}_0 \xrightarrow{\overline{\Upsilon}_0} \left(\overline{\mathbf{R}}_0^{(m), \mathbf{c}} \right)^{\oplus \mu_0} \xrightarrow{\text{pr}_i} \overline{\mathbf{R}}_0^{(m), \mathbf{c}}$$

is equal to $\overline{\Upsilon}_{i,j}$, where pr_i, pr_j are obvious projection maps for every $1 \leq i \leq \mu_0$ and every $1 \leq j \leq \mu_1$. Thus

$$c_{i,j} = H_{\text{sing}}^1(F, \overline{\Upsilon}_{i,j})(\text{AJ}(\text{Sh}'_{\text{sp}})).$$

Let w be a finite place of F . By Lemma 3.4.6, 2.1.3 and Hypothesis 3.2.3, $\mathcal{H}_0 \otimes_{\mathcal{O}_{\lambda}} \mathcal{H}_1$ is pure of weight -1 at w . Thus

$$H^1(F_w, \mathcal{H}_0 \otimes_{\mathcal{O}_{\lambda}} \mathcal{H}_1)$$

vanishes if w is not lying above ℓ , and

$$H_f^1(F_w, \mathcal{H}_0 \otimes_{\mathcal{O}_{\lambda}} \mathcal{H}_1) = H_{\text{st}}^1(F_w, \mathcal{H}_0 \otimes_{\mathcal{O}_{\lambda}} \mathcal{H}_1)$$

if w is lying above ℓ . Then it follows from [NN16, Theorem 5.9] and the proof of [Nek00, Theorem 3.1(ii)] that $\text{AJ}(\text{Sh}'_{\text{sp}})$ is contained in $H_f^1(F, \mathcal{H}_0 \otimes_{\mathcal{O}_{\lambda}} \mathcal{H}_1)$. Hence

$$H^1(F, (1 \otimes \text{pr}_j) \circ (1 \otimes \Upsilon_1))(\text{AJ}(\text{Sh}'_{\text{sp}})) \in H_f^1(F, \mathcal{H}_0),$$

by definition of Bloch–Kato Selmer groups. Therefore, for every finite place w of F not lying above $\Sigma_+^{\Pi_0} \cup \{\mathbf{p}\}$,

$$\text{loc}_w(c_{i,j}) = H^1(F_w, \text{pr}_i \circ \overline{\Upsilon}_0) \left(\text{loc}_w \left(H^1(F, (1 \otimes \text{pr}_j) \circ (1 \otimes \Upsilon_1))(\text{AJ}(\text{Sh}'_{\text{sp}})) \right) \right)$$

is contained in $H_{\text{ns}}^1(F_w, \overline{\mathbf{R}}_0^{(m), \mathbf{c}})$ by [LTX⁺22, Lemma 2.4.3] and the fact that Sh'_{n_0} has good reduction at w .

The claim is proved. $\square$

4. THETA CORRESPONDENCE

In this appendix, we review some results on automorphic representations and theta correspondence that will be useful to us.

Let K be a local or global field of characteristic zero, and let K_1 be an extension field of K with degree at most two. Let $\mathbf{c}$ denote the element in $\text{Gal}(K_1/K)$ with fixed field K . We fix a nontrivial additive character ψ of K (resp. of $K \backslash \mathbf{A}_K$) if K is local (resp. global). For an element $d \in K^{\times}$, let χ_d denote the quadratic character of K (resp. of $K^{\times} \backslash \mathbf{A}_K^{\times}$) corresponding to the quadratic extension $K(\sqrt{d})/K$ via local (resp. global) class field theory when K is local (resp. global).

4.1. The groups. Suppose $\epsilon \in \{\pm 1\}$ is a sign and W is a finite dimensional vector space over K_1 of dimension n equipped with a nondegenerate ϵ -Hermitian c -sesquilinear form

$$\langle -, - \rangle_W : W \times W \rightarrow K_1.$$

We denote by $G(W)$ the group of elements of $GL(W)$ preserving the form $\langle -, - \rangle_W$:

$$G(W)(R) = \{g \in GL(R) : \langle gv, gw \rangle_W = \langle v, w \rangle_W\}.$$

If $K_1 \neq K$ or $\epsilon = 1$, let the discriminant $\text{disc}(W)$ and Hasse–Witt invariant $\epsilon(W)$ of W be normalized as in [Pen25, §2.1]. In particular, if $K_1 = K$ and $\epsilon = 1$, and W has an orthogonal basis $\{v_1, \dots, v_n\}$ with $\langle v_i, v_i \rangle = a_i \in K^\times$ for $1 \leq i \leq n$, then

$$\text{disc}(W) = (-1)^{n(n-1)/2} \prod_{i=1}^n a_i.$$

For notational simplicity, we define $\text{disc}(W) = 1$ and $\epsilon(W) = 1$ if $K_1 = K$ and $\epsilon = -1$. Then the neutral component of $G(W)$ is a reductive group over K . There are several cases to consider:

- (1) If $K_1 = K$ and $\epsilon = 1$, then $G(W) = O(W)$ is an orthogonal group. If $\dim W$ is odd, then $G(W)$ is split (resp. non-quasi-split) if $\epsilon(W) = 1$ (resp. $\epsilon(W) = -1$). If $\dim W$ is even, then G is split if $\text{disc}(W) = 1, \epsilon(W) = 1$, G is non-quasi-split if $\text{disc}(W) = 1, \epsilon(W) = -1$, and G is quasi-split but non-split if $\text{disc}(W) \neq 1$;
- (2) If $K_1 = K$ and $\epsilon = -1$, then $G(W) = \text{Sp}(W)$ is a symplectic group;
- (3) If $K_1 \neq K$, then $G(W) = U(W)$ is a unitary group. $G(W)$ is quasi-split except when $\dim W$ is even and $\epsilon(W) = -1$, in which case it is non-quasi-split.

If $K_1 = K$ and $\epsilon = 1$, the determinant map on $GL(W)$ restricts to a nontrivial quadratic character $\det$ of $G(W) = O(W)$.

If $K_1 = K$ and $\epsilon = -1$, we will consider metaplectic group $\text{Mp}(W)$, which is the unique nonsplit $\mathbb{C}^1$ -covering of $G(W) = \text{Sp}(W)$:

$$1 \rightarrow \mathbb{C}^1 \rightarrow \text{Mp}(W) \rightarrow \text{Sp}(W) \rightarrow 1.$$

Here $\mathbb{C}^1$ is the group of norm-1 elements in $\mathbb{C}^\times$. We can write $\text{Mp}(W) = \text{Sp}(W) \rtimes \mathbb{C}^1$, with multiplication law given by

$$(g_1, z_1) \cdot (g_2, z_2) = (g_1 g_2, z_1 z_2 \cdot c(g_1, g_2))$$

for $g_1, g_2 \in \text{Sp}(W)$ and $z_1, z_2 \in \mathbb{C}^1$, where c is the 2-cocycle of $\text{Sp}(W)$ in $\{\pm 1\}$ given in [RR93]. $\text{Mp}(W)$ has a natural subgroup

$$\widetilde{\text{Sp}}(W) := \text{Sp}(W) \rtimes \{\pm 1\} \subset \text{Mp}(W),$$

which is a nonsplit double cover of $\text{Sp}(W)$. Let $\omega_{W, \psi}$ denote the Weil representation of $\text{Mp}(W)$ with respect to ψ , defined via the Heisenberg group attached to the symplectic space $(W, 2\langle -, - \rangle_W)$. We continue to write $\omega_{W, \psi}$ for its restriction to $\widetilde{\text{Sp}}(W)$. When K is global, we simply write ω_W for the Weil representation.

These classical groups arise naturally in Howe's theory of reductive dual pairs in the symplectic group. We recall some basic facts about these reductive dual pairs and the splitting of the metaplectic cover over them.

Let W be a vector space over K_1 equipped with nondegenerate ϵ -Hermitian c -sesquilinear form

$$\langle -, - \rangle_W : W \times W \rightarrow K_1,$$

and let V be a vector space over K_1 equipped with nondegenerate $(-\epsilon)$ -Hermitian c -sesquilinear form

$$\langle -, - \rangle_V : V \times V \rightarrow K_1.$$

We distinguish the following cases:

- (Case U) $K_1 \neq K$, W is Hermitian and V is skew-Hermitian, or W is skew-Hermitian and V is Hermitian;
- (Case SO1) $K_1 = K$, W is symplectic and V is orthogonal with $\dim V$ odd;
- (Case O1S) $K_1 = K$, W is orthogonal with $\dim V$ odd and V is symplectic;
- (Case O2S) $K_1 = K$, W is orthogonal with $\dim V$ even and V is symplectic;
- (Case SO2) $K_1 = K$, W is symplectic and V is orthogonal with $\dim V$ even.

We collectively refer to Cases O1S and O2S as Case OS, and refer to Cases SO1 and SO2 as Case SO.

Let G and H be algebraic groups over K defined by

$$G = \begin{cases} G(W) & \text{in Cases U, OS, SO2} \\ \widetilde{\mathrm{Sp}}(W) & \text{in Case SO1,} \end{cases} \quad H = \begin{cases} G(V) & \text{in Cases U, O2S, SO} \\ \widetilde{\mathrm{Sp}}(V) & \text{in Case O1S} \end{cases}.$$

Let $\mathbb{W} = W \otimes_{K_1} V$, regarded as a vector space over K and equipped with a symplectic form

$$\mathrm{tr}_{K_1/K}(\langle -, - \rangle_W \otimes_{K_1} \langle -, - \rangle_V).$$

Then $(G(W), H(V))$ is a reductive dual pair in the symplectic group $\mathrm{Sp}(\mathbb{W})$, and there is a natural map

$$\iota : G \times H \rightarrow G(W) \times H(V) \rightarrow \mathrm{Sp}(\mathbb{W}).$$

4.2. Local Gan–Gross–Prasad conjecture. In this subsection, we assume that K is a non-Archimedean local field. We will focus on the group G . Fix a nontrivial additive character ψ of K , and set $m := \dim(W)$.

If G is isomorphic to a metaplectic $\widetilde{\mathrm{Sp}}_{2n}(K)$, then we say an irreducible admissible representation π of $G(K)$ is *genuine* if the nontrivial element in $\ker(\widetilde{\mathrm{Sp}}_{2n}(K) \rightarrow \mathrm{Sp}_{2n}(K))$ acts by -1 . For simplicity, if G is not metaplectic, then every irreducible admissible representation of $G(K)$ is called genuine.

Let $\Pi(G)$ denote the set of all irreducible admissible genuine representations of $G(K)$. Denote by $\Phi(G)$ the set of equivalence classes of representations ϕ of $W_{K_1} \times \mathrm{SL}_2$ of dimension

$$\begin{cases} m-1 & \text{in Case U} \\ m & \text{in Case SO1} \\ m+1 & \text{in Case SO2} \\ m-1 & \text{in Case O1S} \\ m & \text{in Case O2S} \end{cases}$$

which are

$$\begin{cases} \text{conjugate self-dual of sign } (-1)^{m-1} & \text{in Case U} \\ \text{self-dual of sign 1 such that } \det(\phi) = \chi_W & \text{in Case O2S} \\ \text{self-dual of sign 1 such that } \det(\phi) = \mathbf{1} & \text{in Case SO1} \\ \text{self-dual of sign } -\epsilon \text{ such that } \det(\phi) = \mathbf{1} & \text{otherwise} \end{cases}$$

Elements of $\Phi(G)$ are called *L-parameters* for G . We denote by $\Phi_{\mathrm{temp}}(G)$ the subset of equivalence classes of tempered *L-parameters*, that is, the set of $\phi \in \Phi(G)$ such that $\phi(W_E)$ is precompact.

Recall that there is a canonical local Langlands reciprocity map (depending on ψ in the metaplectic case)

$$\mathrm{rec}_W : \Pi(G) \rightarrow \Phi(G);$$

see [GS12, Art13, KMSW14, AG17, CZ21, Ish24]. For any $\pi \in \Pi(G)$, π is tempered if and only if $\mathrm{rec}(\pi)$ is tempered. For $\phi \in \Phi(G)$, we denote by Π_ϕ the inverse image of ϕ , called the *L-packet* of ϕ on G .

We now state the tempered Bessel case of the local Gan–Gross–Prasad conjecture.

Theorem 4.2.1.

(1) Suppose we are in Case U. Set $V_\# := V \oplus L_{(-1)^{\dim V}}$ where $L_{(-1)^{\dim V}}$ is the Hermitian space of dimension 1 and discriminant $(-1)^{\dim V}$. For any $\phi \in \Phi_{\mathrm{temp}}(\mathrm{U}(V))$ and $\phi_\# \in \Phi_{\mathrm{temp}}(\mathrm{U}(V_\#))$, there exists

- a unique pair $(V^\bullet, V_\#^\bullet)$ in which $V^\bullet$ is a Hermitian space over K_1 with $\dim V^\bullet = \dim V$ and $V_\#^\bullet := V^\bullet \oplus L_{(-1)^{\dim V}}$; and
- a pair of irreducible admissible representations $(\pi, \pi_\#) \in \Pi_\phi(\mathrm{U}(V^\bullet)) \times \Pi_{\phi_\#}(\mathrm{U}(V_\#^\bullet))$,

satisfying

$$\mathrm{Hom}_{\mathrm{U}(V^\bullet)}(\pi \otimes \pi_\#, \mathbb{C}) \neq 0.$$

(2) Suppose we are in Case SO. Set $V_\# := V \oplus L_{(-1)^{\dim V+1}}$ where $L_{(-1)^{\dim V+1}}$ is the quadratic space of dimension 1 and discriminant $(-1)^{\dim V+1}$. Let V_0 (resp. V_1) denote the unique even (resp. odd) dimensional element in the set $\{V, V_\#\}$. For $\phi_0 \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_0))$ and $\phi_1 \in \Phi_{\mathrm{temp}}(\mathrm{O}(V_1))$, there exist

- a unique pair $(V_0^\bullet, V_1^\bullet)$ in which $V_0^\bullet$ is a quadratic space with $\dim(V_0^\bullet) = \dim(V_0)$ and $\mathrm{disc}(V_0^\bullet) = \mathrm{disc}(V_0)$ and $V_1^\bullet = V_0^\bullet \oplus L_{(-1)^{\dim V+1}}$; and

- a pair of irreducible admissible representations $(\pi_0, \pi_1) \in \Pi_{\phi_0}(\mathrm{O}(V_0^\bullet)) \times \Pi_{\phi_1}(\mathrm{O}(V_1^\bullet))$, satisfying

$$\mathrm{Hom}_{\mathrm{O}(V_0^\bullet)}(\pi_0 \otimes \pi_1, \mathbb{C}) \neq 0.$$

Proof. Case U is established by Beuzart-Plessis [BP14, BP15, BP16]. Case SO is established in [AG17, Theorem 5.6], extending the Gross–Prasad conjecture in the special orthogonal case established by Waldspurger [Wal10, Wal12, Wal12b, Wal12c]. Note that the assumptions on local Langlands correspondence for orthogonal groups in [AG17, Theorem 5.6] are established in [Art13, Ish24] for odd special orthogonal groups and in [CZ21, Theorem 4.4] for even orthogonal groups. $\square$

4.3. Local theta lifts and Prasad’s conjectures. In this subsection, we assume that K is a local field. We fix a nontrivial additive character ψ of K and a pair of characters $\chi = (\chi_W, \chi_V)$ of $K_1^\times$ such that

- (1) In Case U, $\chi_W|_{K^\times} = \chi_{K_1}^{\dim W}$ and $\chi_V|_{K^\times} = \chi_{K_1}^{\dim V}$;
- (2) In Case SO, χ_W is trivial and $\chi_V = \chi_{\mathrm{disc}(V)}$.
- (3) In Case OS, $\chi_W = \chi_{\mathrm{disc}(W)}$ and χ_V is trivial.

Note that $\chi_W^c = \chi_W^{-1}$ and $\chi_V^c = \chi_V^{-1}$.

Using ψ and χ , the natural map

$$\iota_{W,V} : G \times H \rightarrow \mathrm{Sp}(\mathbb{W})$$

can be lifted to a homomorphism

$$\tilde{\iota}_{W,V,\chi,\psi} \rightarrow \mathrm{Mp}(\mathbb{W});$$

see [Kud94] and [HKS96, §1].

Let $\omega_{\mathbb{W},\psi}$ denote the Weil representation of $\mathrm{Mp}(\mathbb{W})$ with respect to ψ . Using this splitting $\tilde{\iota}_{W,V,\chi,\psi}$, we obtain a representation

$$\omega_{W,V,\psi,\chi} := \omega_{\mathbb{W},\psi} \circ \tilde{\iota}_{W,V,\chi,\psi}$$

of $G \times H$, called the Weil representation of $G \times H$ (with respect to the auxiliary data above).

For any irreducible admissible genuine representation π of $G(K)$, the maximal π -isotypic quotient of $\omega_{W,V,\psi,\chi}$ is of the form

$$\pi \boxtimes \Theta_{W,V,\psi,\chi}(\pi),$$

where $\Theta_{W,V,\psi,\chi}(\pi)$ is either zero or a finite length smooth representation of $H(K)$ [Kud86]. Let $\theta_{W,V,\psi,\chi}(\pi)$ denote the maximal semisimple quotient of $\Theta_{W,V,\psi,\chi}(\pi)$. We have the following standard properties.

Proposition 4.3.1.

- (1) $\theta_{W,V,\psi,\chi}(\pi)$ is either zero or irreducible.
- (2) If K is non-Archimedean and π is supercuspidal, then $\Theta_{W,V,\psi,\chi}(\pi)$ is either zero or irreducible.
- (3) If $K = \mathbb{R}$ and at least one of G and H is compact, then $\Theta_{W,V,\psi,\chi}(\pi)$ is either zero or irreducible.

Proof. The first two follow from the Howe duality conjecture [Kud86, MVW87, How89b, Wal90, GT16]. The third one is due to Howe [How89]. $\square$

The following theorem is known as the local conservation relation (also called the local theta dichotomy); see [HKS96, KR05, Mfm12, GS12, SZ15].

Theorem 4.3.2. Suppose K is non-Archimedean and we are in Case U or SO. If V' is another nondegenerate $(-\epsilon)$ -Hermitian vector space over K_1 with

$$\dim V + \dim V' = 2(\dim W + 2 - [K_1 : K]), \quad \epsilon(V') \neq \epsilon(V)$$

and moreover $\mathrm{disc}(V') = \mathrm{disc}(V)$ in Case SO, then for any irreducible admissible genuine representation π of $G(K)$, exactly one of the two theta lifts $\theta_{W,V,\psi,\chi}(\pi)$ and $\theta_{W,V',\psi,\chi}(\pi)$ is nonzero.

To conclude this subsection, we recall Prasad’s conjectures relating local theta correspondence and local Langlands correspondence.

Theorem 4.3.3. Suppose K is non-Archimedean and π is an irreducible admissible representation of G .

- (1) Suppose we are in Case U and $\dim V = \dim W$. Then there is a unique $(-\epsilon)$ -Hermitian space $V^\bullet$ over K_1 with $\dim(V^\bullet) = \dim(V)$ such that $\theta_{W,V^\bullet,\psi,\chi}(\pi)$ is nonzero. Moreover,

$$\mathrm{rec}_{V^\bullet}(\theta_{W,V^\bullet,\psi,\chi}(\pi)) = \mathrm{rec}_W(\pi) \otimes \chi_V^{-1} \chi_W.$$

(2) Suppose we are in Case U and $\dim V = \dim W + 1$. If $\theta_{W,V^\bullet,\psi,\chi}(\pi)$ is nonzero, then

$$\text{rec}_{V^\bullet}(\theta_{W,V^\bullet,\psi,\chi}(\pi)) = (\text{rec}_W(\pi) \otimes \chi_V^{-1} \chi_W) \oplus \chi_W.$$

(3) Suppose we are in Case U and $\dim V = \dim W - 1$. If $\text{rec}_W(\pi)$ contains χ_V as a subrepresentation, then there is a unique $(-\epsilon)$ -Hermitian space $V^\bullet$ over K_1 with $\dim(V^\bullet) = \dim(V)$ such that $\theta_{W,V^\bullet,\psi,\chi}(\pi)$ is nonzero. Moreover,

$$\text{rec}_W(\pi) = (\text{rec}_{V^\bullet}(\theta_{W,V^\bullet,\psi,\chi}(\pi)) \otimes \chi_W^{-1} \chi_V) \oplus \chi_V$$

(4) Suppose we are in Case SO1 and $\dim V = \dim W + 1$. Then there exists a unique quadratic space $V^\bullet$ over K with $\dim(V^\bullet) = \dim(V)$ and $\text{disc}(V^\bullet) = \text{disc}(V)$ such that $\theta_{W,V^\bullet,\psi,\chi}(\pi)$ is nonzero. Moreover,

$$\text{rec}_{V^\bullet}(\theta_{W,V^\bullet,\psi,\chi}(\pi)) = \text{rec}_W(\pi) \otimes \chi_V.$$

(5) Suppose we are in Case O1S and $\dim V = \dim W - 1$. Then there exists a unique element $\epsilon \in \{\pm 1\}$ such that $\theta_{W,V,\psi,\chi}(\pi \otimes \det^{(1-\epsilon)/2})$ is nonzero. Moreover,

$$\text{rec}_V(\theta_{W,V,\psi,\chi}(\pi \otimes \det^{(1-\epsilon)/2})) = \text{rec}_W(\pi) \otimes \chi_W.$$

(6) Suppose we are in Case SO2 and $\dim V = \dim W + 2$. If $\theta_{W,V,\psi,\chi}(\pi)$ is nonzero, then

$$\text{rec}_V(\theta_{W,V,\psi,\chi}(\pi)) = (\text{rec}_W(\pi) \otimes \chi_V) \oplus \mathbf{1}.$$

Here $\mathbf{1}$ is the trivial representation of W_K .

(7) Suppose we are in Case O2S and $\dim V = \dim W - 2$. If $\text{rec}_W(\pi)$ contains the trivial representation $\mathbf{1}$ as a subrepresentation, then there exists a unique element $\epsilon \in \{\pm 1\}$ such that $\theta_{W,V,\psi,\chi}(\pi \otimes \det^{(1-\epsilon)/2})$ is nonzero. Moreover,

$$\text{rec}_W(\pi) = (\text{rec}_V(\theta_{W,V,\psi,\chi}(\pi \otimes \det^{(1-\epsilon)/2})) \otimes \chi_W) \oplus \mathbf{1}.$$

Proof. (4)-(6) are established by Gan–Ichino [GI16]. (4-5) are established by Gan–Savin [GS12] (cf. [AG17, Theorem B.8]). (6)-(7) are established by Atobe–Gan [AG17, Theorem 4.4]. $\square$

4.4. Global theta lifts. In this subsection, we assume that K is a global field, and set $F := K, F_1 := K_1$. We fix a conjugate self-dual automorphic character μ of $\mathbf{A}_{F_1}$ that satisfying $\mu_u(z) = z/\sqrt{z\bar{z}}$ for every infinite place u of F_1 and $z \in \mathbb{C}^\times$.

If G (resp. H) is isomorphic to a metaplectic group $\widetilde{\text{Sp}}_{2n}$, then the covering $\widetilde{\text{Sp}}_{2n}(F_v) \rightarrow \text{Sp}_{2n}(F_v)$ splits over the hyperspecial maximal compact subgroup $\mathcal{K}_v$ for all but finitely many finite places v of F . So we may regard $\mathcal{K}_v$ as a compact open subgroup of $\widetilde{\text{Sp}}_{2n}(F_v)$. In this case, the restricted tensor product

$$\prod_v {}'G(F_v)$$

with respect to the family $\{\mathcal{K}_v\}_v$ contains $\oplus_v \mu_2$ as a central subgroup. Denote by $G(\mathbf{A}_F)$ (resp. $H(\mathbf{A}_F)$) the quotient of the above restricted tensor product by the central subgroup

$$\{(z_v) \in \bigoplus_v \mu_2 : \prod_v z_v = 1\}.$$

If G (resp. H) is not isomorphic to a metaplectic group, we simply denote by $G(\mathbf{A}_F)$ (resp. $H(\mathbf{A}_F)$) the adelic points of G (resp. H).

Similarly, for all but finitely many finite places v of F , the metaplectic covering $\text{Mp}(\mathbb{W}_v) \rightarrow \text{Sp}(\mathbb{W}_v)$ splits over the hyperspecial maximal compact subgroup $\mathcal{K}_v$. So we may regard $\mathcal{K}_v$ as an open compact subgroup of $\text{Mp}(\mathbb{W}_v)$. Then we define $\text{Mp}(\mathbb{W})(\mathbf{A}_F)$ as the quotient of the restricted tensor product

$$\prod_v {}'\text{Mp}(\mathbb{W}_v)$$

by the central subgroup

$$\{(z_v) \in \prod_v \mathbb{C}^\times : z_v = 1 \text{ for all but finitely many } v, \prod_v z_v = 1\}.$$

The covering $\text{Mp}(\mathbb{W})(\mathbf{A}_F) \rightarrow \text{Sp}(\mathbb{W})(\mathbf{A}_F)$ canonically splits over the subgroup $\text{Sp}(\mathbb{W})(F)$. So we can regard $\text{Sp}(\mathbb{W})(F)$ as a subgroup of $\text{Mp}(\mathbb{W})(\mathbf{A}_F)$.

We fix a convenient set of parameters for the theta correspondence: a nontrivial additive character ψ of $F \backslash \mathbf{A}_F$ and a pair of automorphic characters $\chi = (\chi_W, \chi_V)$ of $K_1 \backslash \mathbf{A}_{K_1}^\times$ such that

- (1) In Case U, $\chi_W = \mu^{(1+(-1)^{\dim W})/2}$ and $\chi_V = \chi^{(1+(-1)^{\dim V})/2}$;
- (2) In Case SO, χ_W is trivial and $\chi_V = \chi_{\text{disc}(V)}$;
- (3) In Case OS, $\chi_W = \chi_{\text{disc}(W)}$ and χ_V is trivial.

Note that $\chi_W^c = \chi_W^{-1}$ and $\chi_V^c = \chi_V^{-1}$. The pair (χ, ψ) fixes a lifting

$$\tilde{\iota}_{W,V} := \bigotimes_v \iota'_{W_v, V_v, \psi_v, \chi_v} : G(\mathbf{A}_F) \times H(\mathbf{A}_F) \rightarrow \text{Mp}(\mathbb{W})(\mathbf{A}_F)$$

of

$$\iota_{W,V} := \bigotimes_v \iota_{W_v, V_v} : G(\mathbf{A}_F) \times H(\mathbf{A}_F) \rightarrow \text{Sp}(\mathbb{W})(\mathbf{A}_F).$$

The global Weil representation $\omega_{\mathbb{W}} := \bigotimes_v \omega'_{\mathbb{W}_v, \psi_v}$ of

$$\prod_v \text{Mp}(\mathbb{W}_v)$$

factors through a representation $\omega_{\mathbb{W}}$ of $\text{Mp}(\mathbb{W})(\mathbf{A}_F)$. Using the lifting $\tilde{\iota}_{W,V}$, we obtain a representation

$$\omega_{W,V} := \omega_{\mathbb{W}} \circ \tilde{\iota}_{W,V}$$

of $G(\mathbf{A}_F) \times H(\mathbf{A}_F)$. If W is skew-Hermitian, we pair it with the 1-dimensional Hermitian space $V' = F_1 e$ with $\|e\| = 1$, and let ω_W denote the restriction of $\omega_{W,V}$ to $G(\mathbf{A}_F)$, called the *Weil representation* of $G(\mathbf{A}_F)$. For each place v of F , we denote by ω_{W_v, ψ_v} the local component of ω_W , which is a representation of $G(F_v)$.

Let $\mathbb{L}$ be a Lagrangian subspace of $\mathbb{W}$. Then the Weil representation $\omega_{W,V}$ is realized on the space of Schwartz functions $\mathcal{S}(\mathbb{L}(\mathbf{A}_F))$. For each Schwartz function $\phi \in \mathcal{S}(\mathbb{L}(\mathbf{A}_F))$, define a theta function on $G(\mathbf{A}_F) \times H(\mathbf{A}_F)$ by

$$\theta_{W,V}(g, h; \phi) := \sum_{x \in \mathbb{L}(F)} \omega_{W,V}(g, h) \phi(x), \quad (g, h) \in G(\mathbf{A}_F) \times H(\mathbf{A}_F).$$

Let $\pi \subset \mathcal{A}_0(G(\mathbf{A}_F))$ be a genuine cuspidal automorphic representation of $G(\mathbf{A}_F)$. Then the *theta lift* $\theta_{W,V}(\pi)$ is defined to be the span of functions on $H(\mathbf{A}_F)$ of the form

$$\theta_{W,V}(\varphi; \phi) : h \mapsto \int_{G(F) \backslash G(\mathbf{A}_F)} \overline{\varphi(g)} \theta_{W,V}(g, h; \phi) dg,$$

for $\varphi \in \pi$ and $\phi \in \mathcal{S}(\mathbb{L}(\mathbf{A}_F))$. Here the measure dg denotes the Tamagawa measure on $G(F) \backslash G(\mathbf{A}_F)$ if G is not metaplectic, and an arbitrary fixed Haar measure otherwise. Since theta functions are of moderate growth and φ is rapidly decreasing, these integrals converge and define automorphic forms on $H(\mathbf{A}_F)$.

We recall the following compatibility property between global and local theta lifts.

Proposition 4.4.1. *Let $\pi \subset \mathcal{A}_0(G(\mathbf{A}_F))$ be a genuine cuspidal automorphic representation of $G(\mathbf{A}_F)$. If $\sigma := \theta_{W,V}(\pi)$ is contained in the space of square-integrable automorphic forms on $H(\mathbf{A}_F)$, then it is irreducible and isomorphic to the restricted tensor product $\bigotimes_v \theta_{W_v, V_v, \psi_v, \chi_v}(\pi_v)$. If moreover σ is cuspidal, then $\pi = \theta_{V,W}(\sigma)$.*

Proof. The first assertion follows from [KR94, Corollary 7.1.3]. The second follows from [GRS93, Proposition 1.2]. $\square$

To end this section, we discuss relation between global theta correspondence and functorial lifts. We first recall the notion of Arthur parameters attached to discrete automorphic representations of orthogonal groups.

Definition 4.4.2. Let π be an automorphic representation of $H(\mathbf{A}_F)$ contained in the space of square-integrable functions on $H(\mathbf{A}_F)$. There is a standard L -homomorphism

$$\xi : {}^L H \rightarrow {}^L (\text{Res}_{F_1/F}(\text{GL}_N)_{F_1}), \quad N = \dim V + \begin{cases} 0 & \text{In Cases U or O1S or SO2} \\ 1 & \text{In Case O2S} \\ -1 & \text{In Case SO1} \end{cases}$$

as defined in [Mok15, §2.1] in Case U (the standard base change embedding) and in [Art13, §1.2] in Case SO or OS. For each finite place v of F such that H_v is unramified, ξ induces a map ξ_* from the set of isomorphism classes of irreducible unramified representations of $\mathrm{SO}(\mathbf{V})(F_v)$ to that of $\mathrm{GL}_N(F_1 \otimes_F F_v)$. A *functorial lift* of π is defined to be an automorphic representation Π of $\mathrm{GL}_N(\mathbf{A}_{F_1})$ that is a finite isobaric sum of discrete automorphic representations such that Π_v is isomorphic to $\mathrm{FL}(\pi_v)$ for all but finitely many finite places v of F such that π_v is unramified. A functorial lift $\mathrm{FL}(\pi)$ exists in Cases O2S, U, O1S, SO1, and SO2; see [Art13], [KMSW14, Theorem 1.7.1], [GI18, Theorem 1.1], [Ish24, Theorem 3.16], [CZ24, Theorem 2.1], respectively. By strong multiplicity one for $\mathrm{GL}_N(\mathbf{A}_{F_1})$ [PS79], this functorial lift is unique up to isomorphism, denoted by $\mathrm{FL}(\pi)$, and we will also call it the *Arthur parameter* of π . To align with the literature, in Case U we also refer to $\mathrm{FL}(\pi)$ as the *base change* of π and write $\mathrm{BC}(\pi)$.

Proposition 4.4.3. *Suppose we are in Cases U or SO. Let $\pi \in \mathcal{A}_0(G(\mathbf{A}_F))$ be a cuspidal automorphic representation of $G(\mathbf{A}_F)$ such that $\theta_{W,V}(\pi)$ is an (irreducible) cuspidal automorphic representation of $H(\mathbf{A}_F)$. Then*

$$\mathrm{FL}(\theta_{W,V}(\pi)) = \begin{cases} \mathrm{BC}(\pi) & \dim V = \dim W \text{ in Case U} \\ (\mathrm{BC}(\pi) \otimes \mu^{-1}) \boxplus \mu^{(1+(-1)^{\dim W})/2} & \dim V = \dim W + 1 \text{ in Case U} \\ \mathrm{FL}(\pi) & \dim V = \dim W + 1 \text{ in Case SO1} \\ (\mathrm{FL}(\pi) \otimes \chi_V) \boxplus \mathbf{1} & \dim V = \dim W + 2 \text{ in Case SO2} \end{cases}.$$

Proof. In Case U, this is [Xue14, Proposition 8.14]. In Case SO, by strong multiplicity one theorem [JS81], it suffices to compare their localizations at finite places v of F where H is split. Thus the assertion follows from Proposition 4.3.3. $\square$

We recall the following criterion of nonvanishing of global theta lifts.

Theorem 4.4.4. *Suppose we are in Case U or SO. Suppose $\dim W = \dim V + 1 - [F_1 : F]$ and $\pi \in \mathcal{A}_0(G(\mathbf{A}_F))$ is a genuine cuspidal automorphic representation of $G(\mathbf{A}_F)$. Assume that $\mathrm{FL}(\pi)_v$ is tempered for every finite place v of F . If $\theta_{\mathbf{W},\mathbf{V}}(\pi)$ is contained in $\mathcal{A}_0(\mathrm{U}(\mathbf{V})(\mathbf{A}_F))$, then it is nonzero if and only if*

- for all places v of F , the local theta lift $\theta_{W_v, V_v, \psi_v, \chi_v}(\pi_v)$ is nonzero, and
- $L(\mathrm{FL}(\pi) \otimes \chi_V; \frac{1}{2})$ is nonzero.

Proof. This follows from [Yam14, Theorem 10.1]. In fact, it is not clear whether the standard L -function $L(s, \pi)$ for π constructed by the doubling method in [Yam14] and the standard L -function of $\mathrm{FL}(\pi)$ coincide. Nevertheless, it follows from Yamana's computation at unramified places [Yam14, Proposition 7.1] that their partial L -functions are equal. It follows from the temperedness assumption and [Yam14, Lemma 7.2] that

$$\mathrm{ord}_{s=\frac{1}{2}} L(s, \pi \otimes \chi_V) = \mathrm{ord}_{s=\frac{1}{2}} L(s, \mathrm{FL}(\pi) \otimes \chi_V).$$

Now [Yam14, Theorem 10.1] applies. $\square$

5. SEESAW AND PROOF OF MAIN THEOREMS

In this section, we use seesaw identities (both local and global) to prove the main theorems. Let r be a positive integer.

5.1. The conjugate self-dual case. Let F be a totally imaginary quadratic extension of a totally real number field F_+ . Let $\mathbf{V}_{2r}$ be a Hermitian space of dimension $2r$ over F , and $\mathbf{V}_1$ be a Hermitian space of dimension 1 over F equipped with an element $e \in \mathbf{V}_1$ satisfying $\|e\| = 1$. Let $\mathbf{W}_{2r}$ be a skew-Hermitian space of dimension $2r$ over F . Set $\mathbf{V}_{2r+1} := \mathbf{V}_{2r} \oplus \mathbf{V}_1$. Let $\iota : \mathrm{U}(\mathbf{V}_{2r}) \subset \mathrm{U}(\mathbf{V}_{2r+1})$ be the natural inclusion. We fix a nontrivial additive character ψ of $F_+ \backslash \mathbf{A}_{F_+}$, and use notations defined in §4.

Consider the inclusion

$$\mathrm{U}(\mathbf{V}_{2r}) \times \mathrm{U}(\mathbf{V}_1) \subset \mathrm{U}(\mathbf{V}_{2r+1})$$

and the diagonal embedding

$$\mathrm{U}(\mathbf{W}_{2r}) \subset \mathrm{U}(\mathbf{W}_{2r}) \times \mathrm{U}(\mathbf{W}_{2r}).$$

$(\mathrm{U}(\mathbf{W}_{2r}), \mathrm{U}(\mathbf{V}_{2r+1}))$ and

$$(\mathrm{U}(\mathbf{W}_{2r}) \times \mathrm{U}(\mathbf{W}_{2r}), \mathrm{U}(\mathbf{V}_{2r}) \times \mathrm{U}(\mathbf{V}_1))$$

are reductive dual pairs. In other words, there is a seesaw diagram:

$$(5.1) \quad \begin{array}{ccc} \mathrm{U}(\mathbf{W}_{2r}) \times \mathrm{U}(\mathbf{W}_{2r}) & & \mathrm{U}(\mathbf{V}_{2r+1}) \\ | & \searrow & | \\ \mathrm{U}(\mathbf{W}_{2r}) & & \mathrm{U}(\mathbf{V}_{2r}) \times \mathrm{U}(\mathbf{V}_1) \end{array}.$$

We fix a conjugate self-dual automorphic character μ of $\mathbf{A}_F$ satisfying $\mu_u(z) = z/\sqrt{z\bar{z}}$ for $z \in \mathbb{C}^\times$ at every infinite place u of F . Then we use the pair (ψ, χ) to define the (both local and global) theta correspondences between the pairs

$$(\mathrm{U}(\mathbf{W}_{2r}), \mathrm{U}(\mathbf{V}_{2r})), \quad (\mathrm{U}(\mathbf{W}_{2r}), \mathrm{U}(\mathbf{V}_1)), \quad (\mathrm{U}(\mathbf{W}_{2r}), \mathrm{U}(\mathbf{V}_{2r+1}))$$

as defined in §4.4. We record the following local seesaw identity attached to the seesaw diagram (5.1).

Lemma 5.1.1. *Let $\mathfrak{p}$ be a finite place of F_+ that is inert in F . For irreducible admissible representations π_0 of $\mathrm{U}(\mathbf{V}_{2r})(F_{+,v})$ and σ_1 of $\mathrm{U}(\mathbf{W}_{2r})(F_{+,v})$, there is a canonical isomorphism*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{U}(\mathbf{W}_{2r})(F_{+,v})}(\Theta_{\mathbf{V}_{2r}, \mathbf{W}_{2r}}(\pi_0) \otimes \omega_{\mathbf{W}_{2r}}, \pi) \\ \cong \mathrm{Hom}_{\mathrm{U}(\mathbf{V}_{2r})(F_{+,v})}(\Theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}}(\sigma_1), \pi_0). \end{aligned}$$

Proof. This is standard. □

We introduce the unitary Gan–Gross–Prasad periods and the Fourier–Jacobi periods.

Definition 5.1.2. Let $\pi_0 \subset \mathcal{A}_0(\mathrm{U}(\mathbf{V}_{2r})(\mathbf{A}_{F_+}))$ and $\pi_1 \subset \mathcal{A}_0(\mathrm{U}(\mathbf{V}_{2r+1})(\mathbf{A}_{F_+}))$ be cuspidal automorphic representations and $f_0 \in \pi_0$ and $f_1 \in \pi_1$ be cusp forms. We define the unitary *Gan–Gross–Prasad period*

$$\mathcal{P}_{\mathrm{GGP}}(f_0, f_1) := \int_{\mathrm{U}(\mathbf{V}_{2r})(F_+) \backslash \mathrm{U}(\mathbf{V}_{2r})(\mathbf{A}_{F_+})} f_0(h) f_1(\iota(h)) dh.$$

Here the measure dh is the Tamagawa measure on $\mathrm{U}(\mathbf{V}_{2r})(\mathbf{A}_{F_+})$. This integral is absolutely convergent since f_0 and f_1 are rapidly decreasing.

We set

$$\mathbb{W}_{2r,1} := \mathbf{W}_{2r} \otimes_{F_+} \mathbf{V}_1, \quad \mathbb{W}_{2r,2r} := \mathbf{W}_{2r} \otimes_{F_+} \mathbf{V}_{2r}, \quad \mathbb{W}_{2r,2r+1} := \mathbf{W}_{2r+1} \otimes_{F_+} \mathbf{V}_{2r+1}.$$

Then they are all symplectic spaces over F_+ as defined in §4.4. Fix Lagrangian subspaces

$$\mathbb{L}_{2r,1} \subset \mathbb{W}_{2r,1}, \quad \mathbb{L}_{2r,2r} \subset \mathbb{W}_{2r,2r},$$

then $\mathbb{L}_{2r,2r} := \mathbb{L}_{2r,2r} \oplus \mathbb{L}_{2r,1}$ is a Lagrangian subspace of $\mathbb{W}_{2r,2r+1}$. For each $n \in \{1, 2r, 2r+1\}$, let $\omega_{\mathbf{W}_{2r}, \mathbf{V}_n}$ denote the Weil representation, which can be realized on the space of Schwartz functions $\mathcal{S}(\mathbb{L}_{2r,n})$. Then

$$\omega_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}} = \omega_{\mathbf{W}_{2r}, \mathbf{V}_{2r}} \hat{\otimes} \omega_{\mathbf{W}_{2r}}.$$

In particular, if $\phi_{2r,2r+1} = \phi_{2r,2r} \otimes \phi_{2r,1} \in \mathcal{S}(\mathbb{L}_{2r,2r}(\mathbf{A}_{F_+})) \otimes \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_{F_+}))$, then

$$\theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}}(g, \iota(h); \phi_{2r,2r+1}) = \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r}}(g, h; \phi_{2r,2r}) \theta_{\mathbf{W}_{2r}, \mathbf{V}_1}(g, \phi_{2r,1})$$

for every $(g, h) \in \mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+}) \times \mathrm{U}(\mathbf{V}_{2r})(\mathbf{A}_{F_+})$.

Definition 5.1.3. Let $\sigma_0, \sigma_1 \subset \mathcal{A}_0(\mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+}))$ be two cuspidal automorphic representations. Let $\varphi_0 \in \sigma_0, \varphi_1 \in \sigma_1$ be automorphic forms and $\phi \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_{F_+}))$ be a Schwartz function. We define the *Fourier–Jacobi period*

$$\mathcal{FJ}(\varphi_0, \varphi_1; \phi) := \int_{\mathrm{U}(\mathbf{W}_{2r})(F_+) \backslash \mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+})} \varphi_0(g) \varphi_1(g) \theta_{\mathbf{W}_{2r}, \mathbf{V}_1}(g; \phi) dg.$$

Here the measure dg is the Tamagawa measure on $\mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+})$. This integral is absolutely convergent since φ_0 and φ_1 are rapidly decreasing and theta functions are of moderate growth.

We will use the following global seesaw identity.

Lemma 5.1.4. *Let $\sigma_1 \subset \mathcal{A}_0(\mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+}))$ and $\pi_0 \subset \mathcal{A}_0(\mathrm{U}(\mathbf{V}_{2r})(\mathbf{A}_{F_+}))$ be cuspidal automorphic representations such that*

$$\sigma_0 = \theta_{\mathbf{V}_{2r}, \mathbf{W}_{2r}}(\bar{\pi}_0)$$

is a cuspidal automorphic representation of $\mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_F)$. Let $\varphi_1 \in \sigma_1$ and $f_0 \in \pi_0$ be cusp forms and $\phi_{2r,1} \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_{F_+}))$, $\phi_{2r,2r} \in \mathcal{S}(\mathbb{L}_{2r,2r}(\mathbf{A}_{F_+}))$ be Schwartz functions. Then

$$\begin{aligned} & \mathcal{FJ}(\theta_{\mathbf{V}_{2r}, \mathbf{W}_{2r}}(\bar{f}_0; \phi_{2r,2r}), \varphi_1; \phi_{2r,1}) \\ &= \mathcal{P}_{\mathrm{GGP}}(f_0, \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}}(\bar{\varphi}_1; \phi_{2r,2r} \otimes \phi_{2r,1})). \end{aligned}$$

Proof. To save space, we write $[\mathrm{U}(\mathbf{V}_{2r})]$ and $[\mathrm{U}(\mathbf{W}_{2r})]$ for

$$\mathrm{U}(\mathbf{V}_{2r})(F_+) \backslash \mathrm{U}(\mathbf{V}_{2r})(\mathbf{A}_{F_+}) \quad \text{and} \quad \mathrm{U}(\mathbf{W}_{2r})(F_+) \backslash \mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+}),$$

respectively. Then

$$\begin{aligned} & \mathcal{FJ}(\theta_{\mathbf{V}_{2r}, \mathbf{W}_{2r}}(\bar{f}_0; \phi_{2r,2r}), \varphi_1; \phi_{2r,1}) \\ &= \int_{[\mathrm{U}(\mathbf{W}_{2r})]} \varphi_1(g) \theta_{\mathbf{W}_{2r}, \mathbf{V}_1}(g; \phi_{2r,1}) \int_{[\mathrm{U}(\mathbf{V}_{2r})]} f_0(h) \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r}}(g, h; \phi_{2r,2r}) dh dg \\ &= \int_{[\mathrm{U}(\mathbf{V}_{2r})]} f_0(h) \int_{[\mathrm{U}(\mathbf{W}_{2r})]} \varphi_1(g) \theta_{\mathbf{W}_{2r}, \mathbf{V}_1}(g; \phi_{2r,1}) \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r}}(g, h; \phi_{2r,2r}) dg dh \\ &= \int_{[\mathrm{U}(\mathbf{V}_{2r})]} f_0(h) \int_{[\mathrm{U}(\mathbf{W}_{2r})]} \varphi_1(g) \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}}(g, \iota(h); \phi_{2r,2r} \otimes \phi_{2r,1}) dg dh \\ &= \mathcal{P}_{\mathrm{GGP}}(f_0, \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}}(\bar{\varphi}_1; \phi_{2r,2r} \otimes \phi_{2r,1})) \end{aligned}$$

□

We now explain how to deduce Theorem B from Theorem D. The key ingredient is the following Burger–Sarnak type principle for Fourier–Jacobi periods on the pair of unitary groups $(\mathrm{U}(\mathbf{W}_{2r}), \mathrm{U}(\mathbf{W}_{2r}))$, in the spirit of [BS91, HL98, Pra07, Zha14]. We first fix notation. For every infinite place u of F , $\mathrm{U}(\mathbf{W}_{2r})(F_{+,u})$ has a maximal compact subgroup $\mathcal{K}_u \cong \mathrm{U}(r) \times \mathrm{U}(r)$. We fix such an isomorphism and denote by $\det_1^{m_1} \det_2^{m_2}$ the character of $\mathcal{K}_u$ defined by

$$(k_1, k_2) \mapsto \det(k_1)^{m_1} \det(k_2)^{m_2}.$$

Proposition 5.1.5. *Assume that $\mathbf{W}_{2r}$ has signature (r, r) at every infinite place. Suppose that*

- (1) Σ is a finite set of places of F_+ containing at least one finite place;
- (2) σ_0 is an automorphic representation of $\mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+})$; and
- (3) $\otimes_{v \in \Sigma} \tau_v$ is an irreducible admissible representation of $\prod_{v \in \Sigma} \mathrm{U}(\mathbf{W}_{2r})(F_{+,v})$ satisfying
 - (a) for every $v \in \Sigma$, the space $\mathrm{Hom}_{\mathrm{U}(\mathbf{W}_{2r})(F_{+,v})}(\sigma_{0,v} \otimes \omega_{\mathbf{W}_{2r},v,\psi_v} \otimes \tau_v, \mathbb{C})$ is nonzero;
 - (b) for every finite place $v \in \Sigma$, τ_v is compactly induced from an irreducible admissible representation ν_v of $Z_v \mathcal{K}_v$, where $\mathcal{K}_v$ is a compact open subgroup of $\mathrm{U}(\mathbf{W}_{2r})(F_{+,v})$ and Z_v is the center of $\mathrm{U}(\mathbf{W}_{2r})(F_{+,v})$; and
 - (c) for every infinite place $u \in \Sigma$, $\tau_u^\vee$ is a holomorphic discrete series that is a generalized Verma module in the sense of [Gar05]. Moreover, if the lowest $\mathcal{K}_u$ -type of $\tau_u^\vee$ is the character $\det_1^{m_1} \det_2^{-m_2}$ for some positive integers m_1, m_2 , then $\sigma_{0,u}$ has lowest $\mathcal{K}_u$ -type $\det_1^{m_1-1} \det_2^{-m_2}$ with multiplicity one.

Then there exists a cuspidal automorphic representation σ_1 of $\mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+})$ satisfying

- (1) for every place $v \in \Sigma$, $\sigma_{1,v}$ is isomorphic to τ_v ; and
- (2) there exist automorphic forms $\varphi_0 \in \sigma_0$, $\varphi_1 \in \sigma_1$ and a Schwartz function $\phi \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_F))$ such that

$$\mathcal{FL}(\varphi_0, \varphi_1; \phi) \neq 0.$$

Proof. The proof is a variant of that of [Zha14, Proposition 2.14]. We write $\mathbf{G} = \mathrm{U}(\mathbf{W}_{2r})$. It follows from the hypothesis that τ_u is induced from its lowest K -type ν_u . We consider the restriction of $\sigma_{0,v} \otimes \omega_{\psi_v, \chi_v}$ to $\mathcal{K}_v$ for each $v \in \Sigma$. By the assumption and Frobenius reciprocity, $\nu^\vee|_{\mathcal{K}_v}$ is a quotient representation of $\sigma_{0,v} \otimes \omega_{\psi_v, \chi_v}|_{\mathcal{K}_v}$. Because $\mathcal{K}_v$ is compact, there exist an automorphic function $\varphi_0 \in \sigma_0$ on $\mathbf{G}(\mathbf{A}_{F_+})$ and a Schwartz function $\phi \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_{F_+}))$ such that the $\prod_{v \in \Sigma} \mathcal{K}_v$ -translates of $f := \varphi_0 \cdot \theta_{\mathbf{W}_{2r}, \mathbf{V}_1}(-; \phi)$ span a $\mathbb{C}$ -vector space that is isomorphic to $\otimes_{v \in \Sigma} \nu^\vee|_{\mathcal{K}_v}$ as representations of $\prod_{v \in \Sigma} \mathcal{K}_v$. We can further assume that

$f(1) \neq 0$. Indeed, $\mathbf{G}(\mathbf{A}_{F_+}^\Sigma)$ acts on the set of all such functions. If they all vanish at the identity element, then they would be identically zero by the weak approximation theorem according to which $\mathbf{G}(F_+)$ is dense in $\mathbf{G}(\mathbf{A}_{F_+, \Sigma})$.

The group $\prod_{v \in \Sigma} Z_v$ acts on f by the character $\prod_{v \in \Sigma} \omega_{\nu_v}^{-1}$, where ω_{ν_v} is the central character of ν_v for every $v \in \Sigma$. Thus the $\prod_{v \in \Sigma} Z_v \mathcal{K}_v$ -translates of f generates a $\mathbb{C}$ -vector space that is isomorphic to $\prod_{v \in \Sigma} \nu_v^{-1}$ as a representation of $\prod_{v \in \Sigma} Z_v \mathcal{K}_v$. As a result, if $v \in \Sigma$ is finite, then the $\mathbf{G}(F_{+,v})$ -translates of f generates a $\mathbb{C}$ -vector space that is isomorphic to $\text{Ind}_{Z_v \mathcal{K}_v}^{\mathbf{G}(F_{+,v})} \nu_v^{-1}$ as representations of $\mathbf{G}(F_{+,v})$. On the other hand, if $u \in \Sigma$ is infinite, then it follows from the relation between the lowest $\mathcal{K}_u$ -type of $\tau_u^\vee$ and $\sigma_{0,u}$ that $U(\mathfrak{g}_u) \rtimes \mathcal{K}_u$ -translates of f generate a $\mathbb{C}$ -vector space that is isomorphic to $\tau_u^\vee$ as a $(\mathfrak{g}_u, \mathcal{K}_u)$ -module (Here $U(\mathfrak{g}_u)$ is the universal enveloping algebra over $\mathbb{C}$ of the Lie algebra of $\mathbf{G}(F_{+,u})$).

Since cusp forms are rapidly decreasing, f is contained in $L^2(\mathbf{G}(F_+) \backslash \mathbf{G}(\mathbf{A}_{F_+}))$. Because the space of automorphic forms are L^2 -dense, one can find an automorphic form φ_1 on $\mathbf{G}(\mathbf{A}_{F_+})$ such that

$$\int_{\mathbf{G}(F_+) \backslash \mathbf{G}(\mathbf{A}_{F_+})} f(g) \varphi_1(g) dg$$

is absolutely convergent and nonzero. Using Hecke projectors and properties of f , we can further assume that Hecke translates of φ_1 generate a cuspidal automorphic representation σ_1 of $\mathbf{G}(\mathbf{A}_{F_+})$ satisfying $\sigma_{1,v} \cong \tau_v$ for every $v \in \Sigma$. Thus $\mathcal{FL}(\varphi_0, \varphi_1; \phi)$ is nonzero.

The theorem is proved. $\square$

We define the notion of admissible places for the coefficient field appearing in Theorem B.

Definition 5.1.6. Let Π be a relevant automorphic representation of $\text{GL}_{2r}(\mathbf{A}_F)$ and E be a strong coefficient field of Π (see Definition 3.1.6). We say that a finite place λ of Σ_E^{fin} , with underlying prime ℓ , is an admissible place (with respect to Π) if the following hold:

- (A1) $\ell \geq 4r + 2$.
- (A2) Σ_+^Π does not contain places over ℓ .
- (A3) The residual representation $\bar{\rho}_{\Pi_0, \lambda}$ is absolutely irreducible. Fix a Gal_F -stable $\mathcal{O}_\lambda$ -lattice $\mathbf{R} \subset \rho_{\Pi, \lambda}(r)$ (which is unique up to homothety), together with an isomorphism $\Xi : \mathbf{R} \xrightarrow{\sim} \mathbf{R}^\vee(1)$.
- (A4-1) Either one of the following two assumptions holds:
 - (a) The image of Gal_F in $\text{GL}(\mathbf{R})$ contains a nontrivial scalar element.
 - (b) $\bar{\mathbf{R}}$ is a semisimple $\kappa_\lambda[\text{Gal}_F]$ -module and $\text{Hom}_{\kappa_\lambda[\text{Gal}_F]}(\text{End}(\bar{\mathbf{R}}), \bar{\mathbf{R}}) = 0$.
- (A4-2) $(\text{GI}_{F', \mathcal{P}, \mathbf{R}}^1)$ from Lemma 2.3.4 holds for $F' = F_{\text{rfx}, +}$ and $\mathcal{P}(T) = T^2 - 1$.
- (A5) The homomorphism $\bar{\rho}_{\Pi, \lambda, +}$ is rigid for $(\Sigma_+^\Pi, \emptyset)$ (see Definition 3.6.1), and $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}_{F(\mu_\ell)}}$ is absolutely irreducible.
- (A6) The composite homomorphism $\mathbb{T}_{2r}^{\Sigma_+^\Pi} \xrightarrow{\phi_\Pi} \mathcal{O}_E \rightarrow \kappa_\lambda$ is cohomologically generic (see Definition 3.1.9 and Definition 3.2.5).

Lemma 5.1.7. Let Π be a relevant automorphic representation of $\text{GL}_{2r}(\mathbf{A}_F)$ and E be a strong coefficient field of Π (see Definition 3.1.6). Suppose $F_+ \neq \mathbb{Q}$ and one of the following two assumptions holds:

- (1) $E = \mathbb{Q}$ and there exists a modular elliptic curve A over F_+ with no complex multiplication over $\bar{F}$ satisfying $\rho_{\Pi, \ell} \cong \text{Sym}^{2r-1} H_{\text{ét}}^1(A_{\bar{F}}, \mathbb{Q}_\ell)|_{\text{Gal}_F}$ for every rational prime ℓ .
- (2) There exists a finite place w of F such that Π_w is supercuspidal; and a good place $\mathfrak{p}$ of F (see Definition 3.3.3) such that $\Pi_{\mathfrak{p}}$ is a Steinberg representation.

Then all but finitely many finite places of E are admissible (with respect to Π).

Proof. We first consider case (1): By [Ser72, Théorème 6] and [Lom15], for sufficiently large rational prime ℓ , the homomorphism

$$\bar{\rho}_{A, \ell}|_{F_{\text{rfx}}} : \text{Gal}_{F_{\text{rfx}}} \rightarrow \text{GL}(H_{\text{ét}}^1(A_{\bar{F}}, \mathbb{F}_\ell))$$

is surjective; let ℓ be such a rational prime. We fix an isomorphism $H_{\text{ét}}^1(A_{\bar{F}}, \mathbb{F}_\ell) \cong \mathbb{F}_\ell^{\oplus 2}$ such that $\rho_{A, \ell}(c)$ is given by the matrix

$$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \in \text{GL}_2(\mathbb{F}_\ell).$$

We need to check that every condition in Definition 5.1.6 excludes only finitely many rational primes ℓ .

For (A1-3) and (A4-1), this is clear.

For (A4-2), we suppose $\ell > 2^{4r-2}$, so

$$\{2^{\pm 1}, 2^{\pm 3}, \dots, 2^{\pm(2r-1)}\}$$

consists of distinct elements in $\mathbb{F}_\ell$, and does not contain $-2 \in \mathbb{F}_\ell$. We take an element $g \in \text{Gal}_{F_{\text{rfx}}}$ whose image under $\bar{\rho}_{A,\ell}$ is

$$\begin{bmatrix} 2 & \\ & 1 \end{bmatrix} \in \text{GL}_2(\mathbb{F}_\ell).$$

Thus $(\text{GI}_{F', \mathcal{P}, \mathbb{R}}^1)$ from Lemma 2.3.4 holds for $F' = F_{\text{rfx},+}$ and $\mathcal{P}(T) = T^2 - 1$ holds by taking the image of gc under $(\bar{\rho}_{\Pi,\ell}, \bar{\varepsilon}_\ell)$.

For (A5), by [LTX+24, Corollary 4.2], the condition that $\bar{\rho}_{\Pi_0, \lambda, +}$ is rigid for $(\Sigma_+^{\min}, \emptyset)$ excludes only finitely many finite places λ of E . The second condition is clearly satisfied.

For (A6), this follows from the same reasoning as in the proof of Lemma 3.8.2.

We now consider case (2): We need to check that every condition in Definition 5.1.6 excludes only finitely many rational primes ℓ .

For (A1) and (A2), this is clear.

For (A3), this follows from [LTX+24, Theorem 4.5.(1)].

For (A4-1), this follows by the same reasoning as in the proof of [LTX+24, Lemma 8.1.4].

For (A4-2), note that, for all but finitely many finite place λ of E strongly disjoint from $\mathfrak{p}$,

$$\{\|\mathfrak{p}\|^{\pm 1} \pmod{\lambda}, \|\mathfrak{p}\|^{\pm 3} \pmod{\lambda}, \dots, \|\mathfrak{p}\|^{\pm(2r-1)} \pmod{\lambda}\}$$

consists of distinct elements and does not contain -1 . For every such λ that also satisfies (A3), the condition $(\text{GI}_{R, F', \mathcal{P}}^1)$ from Lemma 2.3.4 holds for $F' = F_{\text{rfx},+}$ and $\mathcal{P}(T) = T^2 - 1$, by taking the element $(\bar{\rho}_{R, \lambda}, \bar{\varepsilon}_\ell)(\phi_{\mathfrak{p}})$.

For (A5-2), this follows from [LTX+24, Theorem 4.8].

For (A6), this follows from the same reasoning as in the proof of Lemma 3.8.2.

Note that the primes that are excluded can be effectively bounded. $\square$

We now prove Theorem B using the Burger–Sarnak type principle (see Proposition 5.1.5) and seesaw identities.

Theorem 5.1.8. *Let Π_0 be a relevant automorphic representation of $\text{GL}_{2r}(\mathbf{A}_F)$ and E be a strong coefficient field of Π_0 (see Definition 3.1.6). If $L(\frac{1}{2}, \Pi_0) \neq 0$, then for every admissible place λ of E with respect to Π_0 , the Bloch–Kato Selmer group $H_f^1(F, \rho_{\Pi_0, \lambda}(n))$ vanishes.*

Proof. Let λ be an admissible place with underlying rational prime ℓ . We fix an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ that induces the place λ . By (A4-2) and the Chebotarev density theorem, we can find a good inert place $\mathfrak{p}$ of F_+ (see Definition 3.3.3) satisfying

- the underlying prime of $\mathfrak{p}$ is larger than $\max(\ell, 2r+1)$; and
- $\bar{\rho}_{\Pi, \lambda}(\phi_{\mathfrak{p}})$ has generalized eigenvalues $\{\|\mathfrak{p}\| \cdot \alpha_{0,1}^{\pm 1}, \dots, \|\mathfrak{p}\| \cdot \alpha_{0,r}^{\pm 1}\} \subset \overline{\kappa_\lambda}^\times$ with $\alpha_{0,1} = \|\mathfrak{p}\|$ and $\alpha_{0,i} \notin \{\|\mathfrak{p}\|^{\pm 1} \pmod{\lambda}\}$ for every $2 \leq i \leq r$.

We take a $\Pi_{0, \mathfrak{p}}$ -avoiding good representation $\Pi_{1, \mathfrak{p}}^{b, '}$ of $\text{GL}_{2r}(F_{\mathfrak{p}})$ with respect to ι_ℓ (see Definition A.1.2) satisfying

- there exists a lift $F \in W_{F_{\mathfrak{p}}}$ of the arithmetic Frobenius element such that the eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ of $\iota_\ell \text{rec}_{2r}(\Pi_{1, \mathfrak{p}}^{b, '})(F^2)$ are ℓ -adic units; and

•

$$\|\mathfrak{p}\|^2 \notin \{\alpha_i \alpha_j^{-1} | 1 \leq i \neq j \leq 2r\} \cup \{\alpha_i | 1 \leq i \leq 2r\} \subset \overline{\mathbb{F}_\ell}.$$

holds.

Such a representation exists by Lemma A.1.3.

We fix another prime $\mathfrak{q}$ of F_+ inert in F such that $2r\ell$ divides $\|\mathfrak{q}\|^2 - 1$. By Lemma A.1.1, we can take a conjugate-orthogonal supercuspidal representation $\Pi_{1, \mathfrak{q}}^{b, '}$ of $\text{GL}_{2r}(F_{\mathfrak{q}})$ whose associated Galois representation

$$\iota_\ell \text{rec}_{2r}(\Pi_{1, \mathfrak{q}}^{b, '}) : W_{F_{\mathfrak{q}}} \rightarrow \text{GL}_{2r}(\overline{\mathbb{Q}_\ell})$$

is residually absolutely irreducible.

In this paragraph, let v denote a place in $\{\mathfrak{p}, \mathfrak{q}\}$. By the local Gan–Gross–Prasad conjecture (see Theorem 4.2.1(1)), there exists a Hermitian space V'_v of dimension $2r$ over F_v and irreducible admissible representations $\pi'_{0,v}$ and $\pi'_{1,v}$ of $\mathrm{U}(V'_v)$ and $\mathrm{U}(V'_{v,\sharp})$, respectively, satisfying

- (1) $\mathrm{BC}(\pi'_{0,v}) = \Pi_{0,v}$ and $\mathrm{BC}(\pi'_{1,v}) = \Pi_{1,v}^{b,\vee} \boxplus \mathbf{1}$, where $\mathbf{1}$ is the trivial representation of $\mathrm{GL}_1(F_v)$; and
- (2) $\mathrm{Hom}_{\mathrm{U}(V'_v)}(\pi'_{1,v}|_{\mathrm{U}(V'_v)} \otimes \pi'_{0,v}, \mathbb{C}) \neq 0$.

In particular, $\pi'_{1,v}$ is supercuspidal by [MR18, Corollaire 3.5]. By Prasad's conjecture (see Theorem 4.3.3(3)), there exist a unique skew-Hermitian space W'_v of dimension $2r$ over F_v such that the contragredient theta lift

$$\sigma'_{1,v} := (\theta_{V'_{\sharp,v}, W'_v}(\pi'_{1,v}))^\vee$$

is nonzero. Moreover, it follows from Prasad's conjectures (see Theorem 4.3.3) that $\mathrm{BC}(\sigma'_{1,v}) = \Pi_{1,v}^{b,\vee} \otimes \mu_v$. In particular, it follows from [Fin21, Theorem 8.1] and [MR18, Corollaire 3.5] that $\sigma'_{1,v}$ is compactly induced from an irreducible representation of some compact open subgroup of $\mathrm{U}(W'_v)$. Thus it follows from the local seesaw identity (see Lemma 5.1.1) and Proposition 4.3.1 that the theta lift

$$\sigma'_{0,v} := \theta_{V'_{\sharp,v}, W'_v}(\pi'_{0,v})$$

is also nonzero, and

$$\mathrm{Hom}_{\mathrm{U}(W'_v)}(\sigma'_{0,v} \otimes \omega_{W'_v, \psi_v} \otimes \sigma'_{1,v}, \mathbb{C})$$

is nonzero. Moreover, it follows from Prasad's conjectures (see Theorem 4.3.3(1)) that $\mathrm{BC}(\sigma'_{0,v}) = \Pi_{0,v}^\vee$.

We now consider an infinite place u . let W'_u be a skew-Hermitian space of dimension $2r$ and signature (r, r) over $F_{+,u}$, and let V'_u be a Hermitian space of dimension $2r$ and signature $(2r, 0)$ over $F_{+,u}$. Let

$$\sigma'_{1,u} := (\theta_{V'_{u,\sharp}, W'_u}(\mathbf{1}))^\vee$$

be the contragredient of the theta lift of the trivial representation of $\mathrm{U}(V'_{u,\sharp})$ to $\mathrm{U}(W'_u)$, and let

$$\sigma'_{0,u} := \theta_{V'_u, W'_u}(\mathbf{1})$$

be the theta lift of the trivial representation of $\mathrm{U}(V'_u)$ to $\mathrm{U}(W'_u)$. Then it follows from classical calculation (see, for example, [Har07, §2.3] and [Li90]) that

- $\sigma'_{1,u}$ is a holomorphic discrete series representation with Harish-Chandra parameter

$$\tau_1^\vee = \left(\frac{2r+1}{2}, \dots, \frac{3}{2}, -\frac{1}{2}, \dots, -\frac{2r-1}{2} \right)$$

and the lowest $\mathcal{K}_u$ -type being the character $(k_1, k_2) \mapsto \det(k_1)^{r+1} \det(k_2)^{-r}$; and

- $\sigma'_{0,u}$ is a holomorphic discrete series representation with Harish-Chandra parameter

$$\tau_0 = \left(\frac{2r-1}{2}, \dots, -\frac{2r-1}{2} \right)$$

and the lowest $\mathcal{K}_u$ -type being the character $(k_1, k_2) \mapsto \det(k_1)^r \det(k_2)^{-r}$ for every infinite place u of F_+ . In particular, $\sigma'_{1,u}$ is a generalized Verma module for every infinite place u of F_+ (cf. [Gar05]). Moreover, by the local seesaw identity (see Lemma 5.1.1) and Proposition 4.3.1, the space

$$\mathrm{Hom}_{\mathrm{U}(W_u)}(\sigma'_{0,u} \otimes \omega_{W_u} \otimes \sigma'_{1,u}, \mathbb{C})$$

is nonzero for every infinite place u of F_+ . Moreover, $\mathrm{BC}(\sigma'_{0,v}) = \Pi_{0,v}$.

By Arthur's multiplicity formula (see [KMSW14, Theorem 1.7.1]), there exists a skew-Hermitian space $\mathbf{W}_{2r}$ of dimension $2r$ over F with signature (r, r) at every infinite place satisfying $\mathbf{W}_{2r,v} \cong \mathbf{W}'_v$ for every $v \in \{\mathfrak{p}, \mathfrak{q}\}$, and a cuspidal automorphic representation σ_0 of $\mathrm{U}(\mathbf{W}_{2r})$ satisfying $\sigma_{0,v} \cong \sigma'_{0,v}$ for every $v \in \{\mathfrak{p}, \mathfrak{q}\} \cup \Sigma_{F_+}^\infty$ and $\mathrm{BC}(\sigma_0)$ is isomorphic to $\Pi_0^\vee$.

Because $L(\frac{1}{2}, \Pi_0)$ is nonzero, it follows from the local conservation relation (see Theorem 4.3.2), Theorem 2.1.1(1) and Theorem 4.4.4 that there exists a Hermitian space $\mathbf{V}_{2r}$ of dimension $2r$ over F with signature $(2r, 0)$ at every infinite place, such that the conjugate global theta lift

$$\pi_0 := \overline{\theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r}}(\sigma_0)}$$

is an (irreducible) cuspidal automorphic representation of $\mathbf{V}_{2r}(\mathbf{A}_F)$ with trivial Archimedean components. Then it follows from Proposition 4.4.1 and the local conservation relation (see Theorem 4.3.2) that $\mathbf{V}_{2r,v} \cong V'_v$ and $\pi_{0,v} \cong \pi'_{0,v}$, for every $v \in \{\mathfrak{p}, \mathfrak{q}\}$. Moreover, it follows from Lemma 4.4.3 and Proposition 4.4.1 that $\mathrm{BC}(\pi_0)$ is isomorphic to Π_0 , and $\sigma_0 = \theta_{\mathbf{V}_{2r}, \mathbf{W}_{2r}}(\bar{\pi}_0)$. Set $\mathbf{V}_{2r+1} := (\mathbf{V}_{2r})_{\sharp}$.

It follows from the Burger–Sarnak type principle (see Proposition 5.1.5) that there exists a cuspidal automorphic representation σ_1 of $\mathrm{U}(\mathbf{W}_{2r})(\mathbf{A}_{F_+})$ such that $\sigma_{1,v}$ is isomorphic to $\sigma'_{1,v}$ for every $v \in \{\mathfrak{p}, \mathfrak{q}\} \cup \Sigma_{F_+}^\infty$, together with automorphic forms $\varphi_0 \in \sigma_0, \varphi_1 \in \sigma_1$ and a Schwartz function $\phi \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_{F_+}))$ such that

$$\mathcal{FJ}(\varphi_0, \varphi_1; \phi) \neq 0.$$

Set $\Pi_1^b := \mathrm{BC}(\bar{\sigma}_1)$. Then $\Pi_{1,v}^b$ is isomorphic to $\Pi_{1,v}^{b'}$ for every $v \in \{\mathfrak{p}, \mathfrak{q}\}$ and $\Pi_{1,u}^b$ is isomorphic to $\mathrm{BC}(\sigma'_{1,u})$ for every $u \in \Sigma_{F_+}^\infty$. Set $\Pi_1 := (\Pi_1^b \otimes \mu^{-1}) \boxplus \mathbf{1}$, where $\mathbf{1}$ is the trivial representation of $\mathrm{GL}_1(\mathbf{A}_F)$. Then Π_1 is an almost cuspidal relevant representation of $\mathrm{GL}_{2r+1}(\mathbf{A}_F)$ (see Definition 1.1.3).

It follows from the global seesaw identity Lemma 5.1.4 that

$$\pi_1 := \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}}(\bar{\sigma}_1)$$

is nonzero. Because $\mathbf{V}_{2r+1}$ is anisotropic, we know π_1 is an (irreducible) cuspidal automorphic representation of $\mathrm{U}(\mathbf{V}_{2r+1})(\mathbf{A}_{F_+})$. In particular, it follows from Lemma 4.4.3 that π_1 has trivial Archimedean component, and $\pi_{1,v}$ is isomorphic to $\pi'_{1,v}$ for every $v \in \{\mathfrak{p}, \mathfrak{q}\}$. Moreover, it follows from Proposition 4.4.3 that

$$\mathrm{BC}(\pi_1) \cong (\mathrm{BC}(\bar{\sigma}_1) \otimes \mu^{-1}) \boxplus \mathbf{1} = \Pi_1,$$

Thus it follows from the global seesaw identity again that there exist automorphic forms $f_0 \in \pi_0$ and $f_1 \in \pi_1$ such that

$$\mathcal{P}_{\mathrm{GGP}}(f_0, f_1) \neq 0.$$

Let E' be a strong coefficient field of Π_1 containing E . The isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ induces a place λ' of E' with underlying place λ of E . We check that λ' is an admissible place of E with respect to (Π_0, Π_1) (see Definition 3.8.1).

- (L1), (L2), (L4-1) and (L5) are satisfied by $(\Lambda 1)$, $(\Lambda 2)$, $(\Lambda 4-1)$ and $(\Lambda 5)$, respectively.
- For (L3), $\bar{\rho}_{\Pi_0, \lambda'}$ is absolutely irreducible by $(\Lambda 3)$. The restriction of $\rho_{\Pi_1^b, \lambda'} \otimes_{E_{\lambda'}} \overline{\mathbb{Q}_\ell}$ to Gal_{F_q} is residually absolutely irreducible by Proposition 2.1.1 and the definition of $\Pi_{1,q}^{b'}$. Thus $\rho_{\Pi^b, \lambda'}$ is residually absolutely irreducible.
- For (L4-2), it is easy to check that $(\mathrm{GI}_{F_{\mathrm{fix},+}, \mathcal{P}}^1)$ with $\mathcal{P}(T) = T^2 - 1$ is satisfied by taking the element $(\bar{\rho}_{\Pi_0, \lambda', +}, \bar{\rho}_{\Pi_1, \lambda', +}, \bar{\varepsilon}_\ell)(\phi_{\mathfrak{p}})$.
- For (L6), if $\alpha = 0$, then this follows from $(\Lambda 6)$. If $\alpha = 1$, then this follows from the definition of $\Pi_{1,\mathfrak{p}}^{b'}$ and the Chebotarev density theorem applied to the representation $\bar{\rho}_{\Pi_1, \lambda'} \oplus \bar{\varepsilon}_\ell$ of Gal_F , we see that there are infinitely many finite places w of F that are of degree 1 over $\mathbb{Q}$ satisfying that
 - (1) $\Pi_{1,w}$ is unramified with Satake parameter $\{\alpha_{1,1}, \dots, \alpha_{1,2r+1}\}$ in which $\iota_\ell(\alpha_{1,i})$ is an ℓ -adic unit for every $1 \leq i \leq 2r+1$, and
 - (2) $\iota_\ell(\alpha_{1,i}/\alpha_{1,j}) \neq \|w\| \in \overline{\mathbb{K}_{\lambda'}}$ for $1 \leq i \neq j \leq 2r+1$.

Then it follows from [YZ25, Theorem 1.5] that (L6) holds for λ' .

As $F_+ \neq \mathbb{Q}$, Hypothesis 3.2.3 is known for every positive integer $N \geq 2$ by Proposition 3.2.4. We now apply (the proof of) Theorem 3.9.2 to get

$$\mathrm{H}_f^1(F, \rho_{\Pi_0, \lambda}(n)) \otimes_{E_\lambda} E'_{\lambda'} = \mathrm{H}_f^1(F, \rho_{\Pi_0, \lambda'}(r)) = 0.$$

Thus $\mathrm{H}_f^1(F, \rho_{\Pi, \lambda}(n))$ vanishes. □

We now deduce Theorem A and Theorem C from Theorem B.

Corollary 5.1.9. *Let A be a modular elliptic curve over F_+ . Suppose that $F^+ \neq \mathbb{Q}$ and A has no complex multiplication over $\overline{F}$. If the central critical value $L(\mathrm{Sym}^{2r-1} A_F; r)$ does not vanish, then the Bloch–Kato Selmer group*

$$\mathrm{H}_f^1(F, \mathrm{Sym}^{2r-1} \mathrm{H}_{\mathrm{et}}^1(A_{\overline{F}}, \mathbb{Q}_\ell)(r))$$

vanishes for all but finitely many rational primes ℓ .

Proof. By [NT22, Theorem A] and [AC89], $\text{Sym}^{2r-1} A_F$ is modular. Let Π_0 denote the automorphic representation of $\text{GL}_{2r}(\mathbf{A}_F)$ attached to $\text{Sym}^{2r-1} A_F$, which is a cuspidal relevant representation. Thus Π_0 has strong coefficient field $\mathbb{Q}$, and $\rho_{\Pi_0, \ell}$ is conjugate to $\text{Sym}^{2r-1} H_{\text{ét}}^1(A_{\overline{F}}, \mathbb{Q}_{\ell})$ as $\mathbb{Q}_{\ell}[\text{Gal}_F]$ -modules for every rational prime ℓ . Moreover,

$$L\left(\frac{1}{2}, \Pi_0\right) = L\left(r, \text{Sym}^{2r-1} A_F\right).$$

As $F_+ \neq \mathbb{Q}$, Hypothesis 3.2.3 is known for every positive integer $N \geq 2$ by Proposition 3.2.4. Thus the assertion is an immediate consequence of Theorem B and Lemma 5.1.7. $\square$

Corollary 5.1.10. *Let Π be a relevant automorphic representation of $\text{GL}_{2r}(\mathbf{A}_F)$. Suppose that*

- (1) $F_+ \neq \mathbb{Q}$;
 - (2) *There exists a finite place w of F such that Π_w is supercuspidal;*
 - (3) *There exists a good inert place $\mathfrak{p}$ of F (see Definition 3.3.3) such that $\Pi_{\mathfrak{p}}$ is a Steinberg representation.*
- Let E be a strong coefficient field of Π (see Definition 3.1.6). If the central critical value $L(\frac{1}{2}, \Pi)$ does not vanish, then for almost every finite place λ of E , the Bloch–Kato Selmer group $H_f^1(F, \rho_{\Pi, \lambda}(n))$ vanishes.*

Proof. As $F_+ \neq \mathbb{Q}$, Hypothesis 3.2.3 is known for every positive integer $N \geq 2$ by Proposition 3.2.4. Thus the assertion is an immediate consequence of Theorem B and Lemma 5.1.7. $\square$

5.2. The self-dual case. Let F be a totally real number field. Let $\mathbf{V}_{2r+1}$ be a quadratic space of dimension $2r$ over F and let $\mathbf{V}_1$ a quadratic space of dimension 1 over F of discriminant 1. Let $\mathbf{W}_{2r}$ be a symplectic space of dimension $2r$ over F . Set $\mathbf{V}_{2r+2} := \mathbf{V}_{2r+1} \oplus \mathbf{V}_1$. Let $\iota : \text{O}(\mathbf{V}_{2r+1}) \subset \text{O}(\mathbf{V}_{2r+2})$ be the natural inclusion. We fix a nontrivial additive character ψ of $F \backslash \mathbf{A}_F$, and use notation defined in §4.

Consider the inclusion

$$\text{O}(\mathbf{V}_{2r+1}) \times \text{O}(\mathbf{V}_1) \subset \text{O}(\mathbf{V}_{2r+2})$$

and the diagonal embedding

$$\text{Sp}(\mathbf{W}_{2r}) \subset \text{Sp}(\mathbf{W}_{2r}) \times \text{Sp}(\mathbf{W}_{2r}).$$

$(\text{Sp}(\mathbf{W}_{2r}), \text{O}(\mathbf{V}_{2r+2}))$ and

$$(\text{Sp}(\mathbf{W}_{2r}) \times \text{Sp}(\mathbf{W}_{2r}), \text{O}(\mathbf{V}_{2r+1}) \times \text{O}(\mathbf{V}_1))$$

are reductive dual pairs. In other words there is a seesaw diagram:

$$\begin{array}{ccc} \text{Sp}(\mathbf{W}_{2r}) \times \text{Sp}(\mathbf{W}_{2r}) & & \text{O}(\mathbf{V}_{2r+2}) \\ | & \searrow & | \\ \text{Sp}(\mathbf{W}_{2r}) & & \text{O}(\mathbf{V}_{2r+1}) \times \text{O}(\mathbf{V}_1) \end{array}.$$

We record the following local seesaw identity attached to the seesaw diagram (5.1).

Lemma 5.2.1. *For irreducible admissible representations π_0 of $\text{O}(\mathbf{V}_{2r+1})(F_{+,v})$ and σ_1 of $\text{Sp}(\mathbf{W}_{2r})(F_{+,v})$, there is a canonical isomorphism*

$$\begin{aligned} \text{Hom}_{\text{Sp}(\mathbf{W}_{2r})(F_{+,v})}(\Theta_{\mathbf{V}_{2r+1}, \mathbf{W}_{2r}}(\pi_0) \otimes \omega_{\mathbf{W}_{2r}}, \pi) \\ \cong \text{Hom}_{\text{O}(\mathbf{V}_{2r+1})(F_{+,v})}(\Theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+2}}(\sigma_1), \pi_0). \end{aligned}$$

Proof. This is standard. $\square$

We introduce the orthogonal Gross–Prasad periods and the Fourier–Jacobi periods.

Definition 5.2.2. Let $\pi_0 \subset \mathcal{A}_0(\text{O}(\mathbf{V}_{2r+1})(\mathbf{A}_F))$ and $\pi_1 \subset \mathcal{A}_0(\text{O}(\mathbf{V}_{2r+2})(\mathbf{A}_F))$ be cuspidal automorphic representations, and $f_0 \in \pi_0$ and $f_1 \in \pi_1$ be cusp forms. We define the orthogonal *Gross–Prasad period*

$$\mathcal{P}_{\text{GP}}(f_0, f_1) := \int_{\text{O}(\mathbf{V}_{2r+1})(F) \backslash \text{O}(\mathbf{V}_{2r+1})(\mathbf{A}_F)} f_0(h) f_1(\iota(h)) dh.$$

Here the measure dh is the Tamagawa measure on $\text{O}(\mathbf{V}_{2r+1})(\mathbf{A}_F)$. This integral is absolutely convergent since f_0 and f_1 are rapidly decreasing.

We set

$$\mathbb{W}_{2r,1} := \mathbf{W}_{2r} \otimes_F \mathbf{V}_1, \quad \mathbb{W}_{2r,2r+1} := \mathbf{W}_{2r} \otimes_F \mathbf{V}_{2r+1}, \quad \mathbb{W}_{2r,2r+2} := \mathbf{W}_{2r+1} \otimes_F \mathbf{V}_{2r+2}.$$

Then they are all symplectic spaces over F_+ as defined in §4.4. Fix Lagrangian subspaces

$$\mathbb{L}_{2r,1} \subset \mathbb{W}_{2r,1}, \quad \mathbb{L}_{2r,2r+1} \subset \mathbb{W}_{2r,2r+1},$$

then $\mathbb{L}_{2r,2r+1} := \mathbb{L}_{2r,2r+1} \oplus \mathbb{L}_{2r,1}$ is a Lagrangian subspace of $\mathbb{W}_{2r,2r+2}$. For each $n \in \{1, 2r+1, 2r+2\}$, we denote by $\omega_{\mathbf{W}_{2r}, \mathbf{V}_n}$ the Weil representation, which can be realized on the space of Schwartz functions $\mathcal{S}(\mathbb{L}_{2r,n})$. Then

$$\omega_{\mathbf{W}_{2r}, \mathbf{V}_{2r+2}} = \omega_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}} \hat{\otimes} \omega_{\mathbf{W}_{2r}}.$$

In particular, if $\phi_{2r,2r+2} = \phi_{2r,2r+1} \otimes \phi_{2r,1} \in \mathcal{S}(\mathbb{L}_{2r,2r+1}(\mathbf{A}_F)) \otimes \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_F))$, then

$$\theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+2}}(g, \iota(h); \phi_{2r,2r+2}) = \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}}(g, h; \phi_{2r,2r+1}) \theta_{\mathbf{W}_{2r}, \mathbf{V}_1}(g; \phi_{2r,1})$$

for every $(g, h) \in \mathrm{Sp}(\mathbf{W}_{2r})(\mathbf{A}_F) \times \mathrm{O}(\mathbf{V}_{2r+1})(\mathbf{A}_F)$.

Definition 5.2.3. Let $\tilde{\sigma}_0 \subset \mathcal{A}_0(\widetilde{\mathrm{Sp}}(\mathbf{W}_{2r})(\mathbf{A}_F))$ and $\sigma_1 \subset \mathcal{A}_0(\mathrm{Sp}(\mathbf{W}_{2r})(\mathbf{A}_F))$ be genuine cuspidal automorphic representations. Let $\tilde{\varphi}_0 \in \tilde{\sigma}_0$, $\varphi_1 \in \sigma_1$ be cusp forms and $\phi \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_F))$ be a Schwartz function. We define the *Fourier–Jacobi period*

$$\mathcal{FJ}(\tilde{\varphi}_0, \varphi_1; \phi) := \int_{\mathrm{Sp}(\mathbf{W}_{2r})(F) \backslash \mathrm{Sp}(\mathbf{W}_{2r})(\mathbf{A}_F)} \tilde{\varphi}_0(\tilde{g}) \varphi_1(g) \theta_{\mathbf{W}_{2r}, \mathbf{V}_1}(\tilde{g}; \phi) dg.$$

Here $\tilde{g}$ is an arbitrary lift of g to $\widetilde{\mathrm{Sp}}(\mathbf{W}_{2r})$, and the measure dg is the Tamagawa measure on $\mathrm{Sp}(\mathbf{W}_{2r})(\mathbf{A}_F)$. This integral is absolutely convergent since $\tilde{\varphi}_0$ and φ_1 are rapidly decreasing and theta functions are of moderate growth.

We will use the following global seesaw identity.

Lemma 5.2.4. Let $\sigma_1 \subset \mathcal{A}_0(\mathrm{Sp}(\mathbf{W}_{2r})(\mathbf{A}_{F_+}))$ and $\pi_0 \subset \mathcal{A}_0(\mathrm{O}(\mathbf{V}_{2r+1})(\mathbf{A}_{F_+}))$ be cuspidal automorphic representations, such that

$$\tilde{\sigma}_0 = \theta_{\mathbf{V}_{2r+1}, \mathbf{W}_{2r}}(\bar{\pi}_0)$$

is a genuine cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}(\mathbf{W}_{2r})(\mathbf{A}_F)$. Let $\varphi_1 \in \sigma_1$ and $f_0 \in \pi_0$ be cusp forms and $\phi_{2r,1} \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_{F_+}))$, $\phi_{2r,2r+1} \in \mathcal{S}(\mathbb{L}_{2r,2r+1}(\mathbf{A}_{F_+}))$ be Schwartz functions. Then

$$\begin{aligned} \mathcal{FJ}(\theta_{\mathbf{V}_{2r+1}, \mathbf{W}_{2r}}(\bar{f}_0, \phi_{2r,2r+1}), \varphi_1; \phi_{2r,1}) \\ = \mathcal{P}_{\mathrm{GP}}(f_0, \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+2}}(\bar{\varphi}_1; \phi_{2r,2r+1} \otimes \phi_{2r,1})). \end{aligned}$$

Proof. The proof is the same as that of Lemma 5.1.4, thus omitted. $\square$

We now explain how to deduce Conjecture E from Conjecture F. The key ingredient is the following Burger–Sarnak type principle for Fourier–Jacobi periods on the pair $(\mathrm{Sp}(\mathbf{W}_{2r}), \widetilde{\mathrm{Sp}}(\mathbf{W}_{2r}))$, in the spirit of [BS91, HL98, Pra07, Zha14]. We first fix notation. For every infinite place u of F , $\mathrm{Sp}(\mathbf{W}_{2r})(F_u)$ has a maximal compact subgroup $\mathcal{K}_u \cong \mathrm{U}(r)$. Denote the preimage of $\mathcal{K}_u$ in $\widetilde{\mathrm{Sp}}(\mathbf{W}_{2r})(F_u)$ by $\tilde{\mathcal{K}}_u$. Under the identification $\mathcal{K}_u \cong \mathrm{U}(r)$, one can realize

$$\tilde{\mathcal{K}}_u = \{(g, z) | g \in \mathrm{U}(r), z \in \mathbb{C}^\times, \det(g) = z^2\}.$$

There is a genuine “square-root of the determinant” character

$$\sqrt{\det} : \tilde{\mathcal{K}}_u \mapsto \mathbb{C}^\times, (k, z) \mapsto z,$$

which satisfies $(\sqrt{\det})^2 = \det$ (via the projection $\tilde{\mathcal{K}}_u \rightarrow \mathcal{K}_u$).

Proposition 5.2.5. Suppose that

- (1) Σ is a finite set of places of F containing at least one finite place;
- (2) $\tilde{\sigma}_0$ is a genuine automorphic representation of $\widetilde{\mathrm{Sp}}(\mathbf{W}_{2r})(\mathbf{A}_F)$; and
- (3) $\otimes_{v \in \Sigma} \tau_v$ is an irreducible admissible representation of $\prod_{v \in \Sigma} \mathrm{Sp}_{2r}(F_v)$ such that
 - (a) for every $v \in \Sigma$, the space $\mathrm{Hom}_{\mathrm{Sp}(\mathbf{W}_{2r})(F_v)}(\tilde{\sigma}_{0,v} \otimes \omega_{\mathbf{W}_{2r,v}, \psi_v} \otimes \tau_v, \mathbb{C})$ is nonzero;
 - (b) for every finite place $v \in \Sigma$, τ_v is a supercuspidal representation that is compactly induced from a representation of a compact open subgroup $\mathcal{K}_v$ of $\mathrm{Sp}_{2r}(F_v)$; and

- (c) for every infinite place $u \in \Sigma$, $\tau_u^\vee$ is a holomorphic discrete series that is a generalized Verma module in the sense of [Gar05]. Moreover, if the lowest $\mathcal{K}_u$ -type of $\tau_u^\vee$ is the character $\det^m$ for some positive integer m , then $\tilde{\sigma}_{0,u}$ has lowest $\tilde{\mathcal{K}}_u$ -type $(\sqrt{\det})^{2m-1}$ with multiplicity one.

Then there exists a cuspidal automorphic representation σ_1 of $\mathrm{Sp}(\mathbf{W}_{2r})(\mathbf{A}_{F_+})$ satisfying

- (1) for every $v \in \Sigma$, $\sigma_{1,v}$ is isomorphic to τ_v ; and
(2) there exist genuine automorphic forms $\tilde{\varphi}_0 \in \tilde{\sigma}_0$, $\varphi_1 \in \sigma_1$ and a Schwartz function $\phi \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_F))$ such that

$$\mathcal{FL}(\tilde{\varphi}_0, \varphi_1; \phi) \neq 0.$$

Proof. The proof is the same as that of Proposition 5.1.5, thus omitted. $\square$

We introduce the notion of preadmissible places for the coefficient fields appearing in Conjecture F. This is a preliminary notation that can be refined.

Definition 5.2.6. Let A be an elliptic curve over F with $\mathrm{End}(A_{\overline{F}}) = \mathbb{Z}$, and Π a relevant automorphic representation of $\mathrm{GL}_{2r+1}(\mathbf{A}_F)$. Let $E \subset \mathbb{C}$ be a strong coefficient field of Π (Definition 2.2.4). We say that a finite place $\lambda \in \Sigma_E^{\mathrm{fin}}$ with underlying prime ℓ , is preadmissible (with respect to (A, Π)) if

- (pL1) The semi-simplified residual representation $\bar{\rho}_{\Pi, \lambda}$ is either absolutely irreducible or a sum of a self-dual absolutely irreducible representation with a self-dual character.
(pL2) There exists a finite place $\mathfrak{p}$ of F and a finite extension E' of E in $\mathbb{C}$ with a finite place λ' over λ satisfying
(a) $\{\pm 1, \pm \|\mathfrak{p}\|^{\pm 1} \pmod{\ell}, \pm \|\mathfrak{p}\|^{\pm 2} \pmod{\ell}, \dots, \pm \|\mathfrak{p}\|^{\pm 4r} \pmod{\ell}\}$ consists of distinct elements;
(b) E has good reduction at $\mathfrak{p}$ with $a_{\mathfrak{p}}(E) - (\|\mathfrak{p}\| + 1)$ is divisible by ℓ ;
(c) $\Pi_{\mathfrak{p}}$ is unramified with Satake parameter $\{-1, \alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}\} \subset \mathcal{O}_{E'}$ such that $\{\alpha_i | 1 \leq i \leq r\}$ is disjoint from $\{\pm 1, \pm \|\mathfrak{p}\|^{\pm 1}, \dots, \pm \|\mathfrak{p}\|^{\pm 4r}\}$ in $\kappa_{\lambda'}$.
(pL3) There exists a finite place v of F such that Π_v is unramified with Satake parameters $\{1, \alpha_{v,1}^{\pm 1}, \dots, \alpha_{v,r}^{\pm 1}\}$ satisfying

$$\|v\| \notin \{\alpha_{v,i} \alpha_{v,j}^{-1} | 1 \leq i \neq j \leq r\} \cup \{\alpha_{v,i}^{\pm 1} | 1 \leq i \leq r\} \cup \{1\} \subset \overline{\mathbb{F}}_{\ell}.$$

We give an example where it is known that all but finitely many finite places λ of E are admissible.

Lemma 5.2.7. Let A and Π be as in Definition 5.2.6. If we assume that there exist finite places $\mathfrak{p}, \mathfrak{q}$ of F such that

- (1) A has split multiplicative reduction at $\mathfrak{p}$, and
(2) $\Pi_{\mathfrak{p}}$ is unramified with Satake parameter of the form $\{-1, \alpha_1^{\pm 1}, \dots, \alpha_r^{\pm 1}\}$ satisfying $\alpha_i \neq \pm 1$ for every $1 \leq i \leq r$; and
(3) $\Pi_{\mathfrak{q}}$ is either supercuspidal or an isobaric sum of a self-dual supercuspidal representation with a self-dual character.

then there exists an effective constant $N(F, A, \Pi_{\mathfrak{p}}, \Pi_{\mathfrak{q}})$ depending on $F, A, \Pi_{\mathfrak{p}}$, and $\Pi_{\mathfrak{q}}$ such that every finite place λ of E with underlying prime ℓ greater than $N(F, A, \Pi_{\mathfrak{p}}, \Pi_{\mathfrak{q}})$ is admissible with respect to (A, Π) .

Proof. We show that every condition in Definition 5.2.6 excludes only finitely many finite places of E .

For (pL1), the condition $\bar{\rho}_{\Pi_1^{\mathfrak{p}}, \lambda}$ is absolutely irreducible only excludes finitely many finite places λ of E by [LTX⁺24, Theorem 4.5.(1)] and (3).

For (pL2), it follows from temperedness (see Proposition 2.2.2) that $|\alpha_i| = 1$ for every $1 \leq i \leq r$. Moreover, α_i is an algebraic number for every $1 \leq i \leq r$ by Remark 2.2.5. Thus it is clear that when ℓ is large, (pL2)(a, c) is satisfied. By the Chebotarev density theorem, (L2) is satisfied for all such ℓ .

For (pL3), it follows from Proposition 2.2.2(1) that $\|\alpha_i\| = 1$ for every $1 \leq i \leq 2r+1$. Thus, for every sufficiently large rational prime ℓ ,

$$\|\mathfrak{p}\| \notin \{\pm \alpha_i \alpha_j^{-1} | 1 \leq i \neq j \leq r\} \cup \{\pm \alpha_i^{\pm 1} | 1 \leq i \leq r\} \cup \{\pm 1\} \subset \overline{\mathbb{F}}_{\ell}.$$

Suppose λ is a finite place of E over ℓ , we fix an isomorphism $\iota_{\ell} : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_{\ell}$ which induces λ . Applying the Chebotarev density theorem to the representation $\bar{\rho}_{\Pi, \lambda} \oplus \bar{\varepsilon}_{\ell}$ of Gal_F , we see that (L3) holds for λ . $\square$

Theorem 5.2.8. Let r be a positive integer, and let A be a modular elliptic curve over F with no complex multiplication over $\overline{F}$. Assume Conjecture F holds for r and A . If the central critical value $L(\mathrm{Sym}^{2r-1} A; r)$

does not vanish, then there is an effective constant $N(F, A, r)$ depending only on F, A , and r such that the Bloch–Kato Selmer group

$$H_f^1(F, \text{Sym}^{2r-1} H_{\text{ét}}^1(A; \mathbb{Q}_\ell)(r)) .$$

vanishes for all rational primes ℓ greater than $N(F, A, r)$.

Proof. By [NT22, Theorem A], $\text{Sym}^{2r-1} A$ is modular. Let Π_0 denote the automorphic representation of $\text{GL}_{2r}(\mathbf{A}_F)$ attached to $\text{Sym}^{2r-1} A$, which is a cuspidal automorphic representation. Thus Π_0 has strong coefficient field $\mathbb{Q}$, and $\rho_{\Pi_0, \ell}$ is conjugate to $\text{Sym}^{2r-1} H_{\text{ét}}^1(A_{\overline{F}}, \mathbb{Q}_\ell)$ as $\mathbb{Q}_\ell[\text{Gal}_F]$ -modules for every rational prime ℓ . Moreover,

$$L\left(\frac{1}{2}, \Pi_0\right) = L\left(r, \text{Sym}^{2r-1} A\right) .$$

By [Ser72, Théorème 6] and [Lom15], there is an effective constant $N_1(F, A)$ depending only on A such that the homomorphism

$$\bar{\rho}_{A, \ell} : \text{Gal}_F \rightarrow \text{GL}\left(H_{\text{ét}}^1(A_{\overline{F}}, \mathbb{F}_\ell)\right)$$

is surjective for every rational prime ℓ greater than $N_1(F, A)$.

Suppose there is an effective constant $N_2(F, A, r)$ such that Conjecture F holds for any preadmissible finite place λ of the strong coefficient field with underlying rational prime $\ell > N_2(F, A, r)$. We set

$$N(F, A, r) := \max(N_1(F, A), N_2(F, A, r), 2^{16r}).$$

Let $\ell > N(F, A, r)$ be a rational prime with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. Then we know the set

$$B_2 := \{\pm 1, \pm 2^{\pm 1}, \pm 2^{\pm 2}, \dots, \pm 2^{\pm 4r}\}$$

consists of distinct elements in $\mathbb{F}_\ell$. By the Chebotarev density theorem, we can find a finite place $\mathfrak{p}$ of F satisfying

- The rational prime p underlying $\mathfrak{p}$ is larger than $\max(\ell, 2r)$;
- A has good reduction at $\mathfrak{p}$; and
- $\bar{\rho}_{A, \ell}(\phi_{\mathfrak{p}})$ has eigenvalues $\{2, 1\}$.

We fix a totally positive element $\mathfrak{d} \in F^\times$ satisfying $(-1)^{r+1} \mathfrak{d} \neq 1 \in F^\times / (F^\times)^2$.

We take a supercuspidal B_2 -avoiding good representation $\Pi_{1, \mathfrak{p}}^{bb, \iota}$ of $\text{GL}_{2r}(F_{\mathfrak{p}})$ with respect to ι_ℓ (see Definition A.2.1) satisfying

- $\iota_\ell \text{rec}_{2r}(\Pi_{1, \mathfrak{p}}^{bb, \iota})$ is residually absolutely irreducible; and
- there exists a lift $F \in W_{F_{\mathfrak{p}}}$ of the arithmetic Frobenius element such that the eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ of $\iota_\ell \text{rec}_{2r}(\Pi_{1, \mathfrak{p}}^{bb, \iota})(F)$ are ℓ -adic units; and
-

$$\|\mathfrak{p}\| \notin \{\pm \alpha_i \alpha_j^{-1} | 1 \leq i \neq j \leq 2r\} \cup \{\pm \alpha_i | 1 \leq i \leq 2r\} \subset \overline{\mathbb{F}_\ell}.$$

holds.

Such a representation exists by Lemma A.2.2. Set $\Pi_{1, \mathfrak{p}}^{b, \iota} := \Pi_{1, \mathfrak{p}}^{bb, \iota} \boxplus \chi_{(-1)^{r+1} \mathfrak{d}}$.

By the local Gan–Gross–Prasad conjecture (see Theorem 4.2.1(2)), there exists a quadratic space $V_{\mathfrak{p}}'$ of dimension $2r + 1$ over $F_{\mathfrak{p}}$ with $\text{disc}(V_{\mathfrak{p}}') = (-1)^r \mathfrak{d} \in F_{\mathfrak{p}}^\times / (F_{\mathfrak{p}}^\times)^2$; and irreducible admissible representations $\pi'_{0, \mathfrak{p}}$ and $\pi'_{1, \mathfrak{p}}$ of $\text{O}(V_{\mathfrak{p}}')$ and $\text{O}(V_{\mathfrak{p}, \sharp}')$, respectively, satisfying

- (1) $\text{FL}(\pi'_{0, \mathfrak{p}}) = \Pi_{0, \mathfrak{p}}$ and $\text{FL}(\pi'_{1, \mathfrak{p}}) = \Pi_{1, v}^{b, \iota} \boxplus \mathbf{1}$, where $\mathbf{1}$ is the trivial representation of $\text{GL}_1(F_{\mathfrak{p}})$; and
- (2) $\text{Hom}_{\text{O}(V_{\mathfrak{p}}')}(\pi'_{1, \mathfrak{p}}|_{\text{O}(V_{\mathfrak{p}}')} \otimes \pi'_{0, \mathfrak{p}}, \mathbb{C}) \neq 0$.

In particular, $\pi'_{1, \mathfrak{p}}$ is supercuspidal by [MR18, Corollaire 3.5]. According to Prasad’s conjecture (see Theorem 4.3.3(7)), upon changing $(\pi'_{0, \mathfrak{p}}, \pi'_{1, \mathfrak{p}})$ to $(\pi'_{0, \mathfrak{p}} \otimes \det, \pi'_{1, \mathfrak{p}} \otimes \det)$ if necessary, we can assume that the contragredient theta lift

$$\sigma'_{1, \mathfrak{p}} := (\theta_{V_{\mathfrak{p}, \sharp}', W_{\mathfrak{p}}}(\pi'_{1, \mathfrak{p}}))^{\vee}$$

is nonzero, where $W_{\mathfrak{p}}$ is a symplectic space of dimension $2r$ over $F_{\mathfrak{p}}$. Moreover, it follows from Prasad’s conjectures (see Theorem 4.3.3) that $\text{FL}(\sigma'_{1, \mathfrak{p}}) = \Pi_{1, \mathfrak{p}}^{b, \iota} \otimes \chi_{(-1)^{r+1} \mathfrak{d}}$. In particular, it follows from [Fin21, Theorem 8.1] and [MR18, Corollaire 3.5] that $\sigma'_{1, \mathfrak{p}}$ is supercuspidal and compactly induced from an irreducible

representation of some compact open subgroup of $\mathrm{Sp}(W_{\mathfrak{p}})$. By the local seesaw identity (see Lemma 5.2.1) and Proposition 4.3.1, the theta lift

$$\tilde{\sigma}'_{0,\mathfrak{p}} := \theta_{V'_{\mathfrak{p}}, W'_{\mathfrak{p}}}(\pi'_{0,\mathfrak{p}}{}^{\vee})$$

is also nonzero, and

$$\mathrm{Hom}_{\mathrm{Sp}(W_{\mathfrak{p}})}(\tilde{\sigma}'_{0,\mathfrak{p}} \otimes \omega_{W'_{\mathfrak{p}}, \psi_{\mathfrak{p}}} \otimes \sigma'_{1,\mathfrak{p}}, \mathbb{C})$$

is nonzero. Moreover, it follows from Prasad's conjecture (see Theorem 4.3.3(5)) that

$$\mathrm{FL}(\tilde{\sigma}'_{0,\mathfrak{p}}) = \Pi_{0,\mathfrak{p}}^{\vee} \otimes \chi_{(-1)^r \mathfrak{d}}.$$

We now consider an infinite place u . Let W_u be a symplectic space of dimension $2r$ over F_u , and let V'_u be a quadratic space of dimension $2r+1$ over F_u with signature $(2r+1, 0)$. Let

$$\sigma'_{1,u} := (\theta_{V'_{u,\sharp}, W_u}(\mathbf{1}))^{\vee}$$

be the contragredient of the theta lift of the trivial representation of $\mathrm{O}(V'_{u,\sharp})$ to $\mathrm{Sp}(W'_u)$, and let

$$\tilde{\sigma}'_{0,u} := \theta_{V'_u, W_u}(\mathbf{1})$$

be the theta lift of the trivial representation of $\mathrm{O}(V'_u)$ to $\widetilde{\mathrm{Sp}}(W_u)$. Then it follows from classical calculation (see, for example, [KR90, Proposition 2.1] and [AB98, Theorem 3.3]) that

- $\sigma'_{1,u}{}^{\vee}$ is a holomorphic discrete series representation with Harish-Chandra parameter

$$\tau_1^{\vee} = (r, r-1, \dots, 1)$$

and the lowest $\mathcal{K}_u$ -type being the character $\det^{r+1}$; and

- $\tilde{\sigma}'_{0,u}$ is a holomorphic discrete series representation with Harish-Chandra parameter

$$\tilde{\tau}_0 = \left(\frac{2r-1}{2}, \frac{2r-3}{2}, \dots, \frac{1}{2} \right)$$

and the lowest $\tilde{\mathcal{K}}_u$ -type being the character $\sqrt{\det}^{2r+1}$ for every infinite place u of F . In particular, $\sigma'_{1,u}{}^{\vee}$ is a generalized Verma module for every infinite place u of F (cf. [Gar05]). Moreover, by the local seesaw identity (see Lemma 5.2.1) and Proposition 4.3.1, the space

$$\mathrm{Hom}_{\mathrm{Sp}(W_u)}(\tilde{\sigma}'_{0,u} \otimes \omega_{W_u} \otimes \sigma'_{1,u}, \mathbb{C})$$

is nonzero for every infinite place u of F . Moreover, $\mathrm{FL}(\sigma'_{0,v}) = \Pi_{0,u} \otimes \chi_{(-1)^r}$.

By Arthur's multiplicity formula [GI18, Theorem 1.4], there exists a genuine cuspidal automorphic representation $\tilde{\sigma}_0$ of $\widetilde{\mathrm{Sp}}(\mathbf{W}_{2r})$ satisfying $\tilde{\sigma}_{0,u} \cong \tilde{\sigma}'_{0,u}$ for every $u \in \Sigma_{F_+}^{\infty} \cup \{\mathfrak{p}\}$ and $\mathrm{FL}(\tilde{\sigma}_0) \cong \Pi_0^{\vee} \otimes \chi_{(-1)^r \mathfrak{d}}$.

Because $L(\frac{1}{2}, \Pi_0)$ is nonzero, it follows from the local conservation relation (see Theorem 4.3.2), Theorem 2.2.2(1) and Theorem 4.4.4 that there exists a unique quadratic space $\mathbf{V}_{2r+1}$ of dimension $2r+1$ over F satisfying

- $\mathbf{V}_{2r+1}$ has signature $(2r+1, 0)$ at every infinite place of F ;
- $\mathrm{disc}(\mathbf{V}_{2r+1}) = (-1)^r \mathfrak{d} \in F^{\times}/(F^{\times})^2$; and
- the conjugate global theta lift

$$\pi_0 := \overline{\theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+1}}(\tilde{\sigma}_0)}$$

is an (irreducible) cuspidal automorphic representation of $\mathbf{V}_{2r+1}(\mathbf{A}_F)$ with trivial Archimedean components.

Then it follows from Proposition 4.4.1 and the local conservation relation (see Theorem 4.3.2) that $\mathbf{V}_{2r+1,\mathfrak{p}}$ is isomorphic to $V'_{\mathfrak{p}}$ and $\pi_{0,\mathfrak{p}}$ is isomorphic to $\pi'_{0,\mathfrak{p}}$. Moreover, it follows from Lemma 4.4.3 and Proposition 4.4.1 that $\mathrm{FL}(\pi_0)$ is isomorphic to Π_0 , and

$$\tilde{\sigma}_0 = \theta_{\mathbf{V}_{2r+1}, \mathbf{W}_{2r}}(\bar{\pi}_0).$$

Set $\mathbf{V}_{2r+2} := (\mathbf{V}_{2r+1})_{\sharp}$.

It follows from the Burger-Sarnak type principle (see Proposition 5.1.5) that there exists a cuspidal automorphic representation σ_1 of $\mathrm{Sp}(\mathbf{W}_{2r})(\mathbf{A}_{F_+})$ such that $\sigma_{1,v}$ is isomorphic to $\sigma'_{1,v}$ for every $v \in \{\mathfrak{p}, \mathfrak{q}\} \cup \Sigma_{F_+}^{\infty}$; together with automorphic forms $\tilde{\varphi}_0 \in \tilde{\sigma}_0$, $\varphi_1 \in \sigma_1$ and a Schwartz function $\phi \in \mathcal{S}(\mathbb{L}_{2r,1}(\mathbf{A}_{F_+}))$ such that

$$\mathcal{FJ}(\tilde{\varphi}_0, \varphi_1; \phi) \neq 0.$$

Set $\Pi_1^b := \text{FL}(\bar{\sigma}_1)$, which satisfies $\Pi_{1,\mathfrak{p}}^b \cong \Pi_{1,\mathfrak{p}}^{b,\prime} \otimes \chi_{(-1)^{r+1}\mathfrak{d}}$ and $\Pi_{1,u}^b \cong \text{FL}(\sigma_{1,u}'^{\vee}) \otimes \chi_{(-1)^{r+1}}$ for every $u \in \Sigma_F^\infty$. Then $\Pi := \Pi_1^b \otimes \chi_{(-1)^{r+1}\mathfrak{d}}$ is a relevant automorphic representation of $\text{GL}_{2r+1}(\mathbf{A}_F)$ (see Definition 2.2.1). Set $\Pi_1 := \Pi \boxplus \mathbf{1}$, where $\mathbf{1}$ is the trivial representation of $\text{GL}_1(\mathbf{A}_F)$.

It follows from the global seesaw identity (see Lemma 5.1.4) that

$$\pi_1 := \theta_{\mathbf{W}_{2r}, \mathbf{V}_{2r+2}}(\bar{\sigma}_1)$$

is nonzero. Because $\mathbf{V}_{2r+2}$ is anisotropic, we know π_1 is an (irreducible) cuspidal automorphic representation of $\text{O}(\mathbf{V}_{2r+2})(\mathbf{A}_F)$. In particular, it follows from Lemma 4.4.3 that π_1 has trivial Archimedean component, and $\pi_{1,\mathfrak{p}}$ is isomorphic to $\pi_{1,\mathfrak{p}}'$. Moreover, it follows from Proposition 4.4.3 that

$$\text{FL}(\pi_1) \cong (\text{FL}(\bar{\sigma}_1) \otimes \chi_{(-1)^{r+1}\mathfrak{d}}) \boxplus \mathbf{1} = \Pi_1.$$

Thus it follows from the global seesaw identity again that there exist automorphic forms $f_0 \in \pi_0$ and $f_1 \in \pi_1$ such that

$$\mathcal{P}_{\text{GP}}(f_0, f_1) \neq 0.$$

Let $E \subset \mathbb{C}$ be a strong coefficient field of Π . The isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ induces a place λ of E . We check that λ is a preadmissible place of E with respect to (E, Π_1) (see Definition 5.2.6).

- For (pL1), note that the restriction of $\rho_{\Pi,\lambda}$ to Gal_{F_q} is a direct sum of a residually absolutely irreducible self-dual representation σ with a self-dual character χ by Proposition 2.2.2 and the definition of $\Pi_{1,\mathfrak{p}}^{b,\prime}$. If the semi-simplified residual representation $\bar{\rho}_{\Pi,\lambda}$ is not irreducible, then it is a sum of a self-dual absolutely irreducible representation with a self-dual character. On the other hand, if it is irreducible, then it is absolutely irreducible, because otherwise $\bar{\rho}_{\Pi,\lambda} \otimes_{\kappa_\lambda} \bar{\kappa}_\lambda$ is a sum of several irreducible representations of the same dimension, contradicting the fact that $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}_{F_q}} = \bar{\sigma} \oplus \bar{\chi}$.
- (pL2) holds by our choice of $\mathfrak{p}$ and the definition of $\Pi_{1,\mathfrak{p}}^{b,\prime}$; see Definition A.2.1.
- (pL3) holds by the definition of $\Pi_{1,\mathfrak{p}}^{b,\prime}$ and the Chebotarev density theorem applied to the representation $\bar{\rho}_{\Pi,\lambda} \oplus \bar{\varepsilon}_\ell$ of Gal_F .

We now apply Conjecture F to the preadmissible place λ to get

$$\text{H}_f^1(F, \text{Sym}^{2r-1} \text{H}_{\text{ét}}^1(A; \mathbb{Q}_\ell)(r)) \otimes_{\mathbb{Q}_\ell} E_\lambda = \text{H}_f^1(F, \rho_{\Pi_0,\ell}(r)) \otimes_{\mathbb{Q}_\ell} E_\lambda = \text{H}_f^1(F, \rho_{\Pi_0,\lambda}(r)) = 0.$$

Thus $\text{H}_f^1(F, \text{Sym}^{2r-1} \text{H}_{\text{ét}}^1(A; \mathbb{Q}_\ell)(r))$ vanishes.

The theorem is proved. $\square$

APPENDIX A. POLARIZED LOCAL GALOIS REPRESENTATIONS

In this appendix, we construct certain (conjugate) self-dual local Galois representations of special kind. These representations will be used in the Burger–Sarnak type principles.

A.1. Special conjugate self-dual local Galois representations. In this subsection, we construct certain conjugate self-dual local Galois representations of special kind.

Let p be an odd rational prime, and let K be a finite extension of $\mathbb{Q}_p$. Denote by κ the residue field of K , of cardinality q . Let K_1 be the unramified quadratic extension of K . Let $\mathcal{O}_K$ (resp. $\mathcal{O}_{K_1}$) denote the ring of integers of K (resp. K_1) with maximal ideal $\mathfrak{m}_K$ (resp. $\mathfrak{m}_{K_1}$). Denote by κ_1 the residue field of K_1 . Fix a uniformizer ϖ_K of K .

We care about representations of W_{K_1} that are conjugate-orthogonal, that is, if we write $\Pi^\theta := (\Pi^s)^\vee$, where Π^s is the conjugate of Π by an element $s \in W_K$ which maps to $\mathfrak{c} \in \text{Gal}(K_1/K)$, then there is an isomorphism $f : \Pi^\theta \xrightarrow{\sim} \Pi$ satisfying $(f^\vee)^s = f$. Constructing irreducible conjugate self-dual representations of W_{K_1} is more complicated than expected. We will only provide the following construction of residually absolutely irreducible conjugate-orthogonal representations when there is a tamely ramified cyclic extension of degree $2r$.

Lemma A.1.1. *Let ℓ be a rational prime distinct from p , with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. Suppose ℓ is coprime to $2pr$ and $2r|(q^2 - 1)$. Then there exists a conjugate-orthogonal supercuspidal representation Π of $\text{GL}_{2r}(K_1)$ such that the Galois representation*

$$\iota_\ell \text{rec}_{2r}(\Pi) : W_{K_1} \rightarrow \text{GL}_{2r}(\overline{\mathbb{Q}_\ell})$$

attached to Π via local Langlands correspondence is residually absolutely irreducible.

Proof. By local Langlands correspondence for $\mathrm{GL}_{2r}(K_1)$, it suffices to construct a residually absolutely irreducible $2r$ -dimensional representation (ρ, V) of W_{K_1} with $\overline{\mathbb{Q}_\ell}$ -coefficients, a lift $s \in W_K$ of $\mathfrak{c} \in \mathrm{Gal}(K_1/K)$, and a nondegenerate pairing $\langle -, - \rangle : V \times V \rightarrow \overline{\mathbb{Q}_\ell}$ satisfying

$$(A.1) \quad \begin{cases} \langle \rho(\tau)f, \rho(s\tau s^{-1})g \rangle = \langle f, g \rangle \\ \langle g, f \rangle = \langle f, \rho(s^2)g \rangle \end{cases}$$

for all $\tau \in W_{K_1}$ and $f, g \in V$.

Let $\bar{\gamma} \in \kappa_1^\times$ be such that $\{\bar{\gamma}^q, \bar{\gamma}\}$ is a κ -basis of κ_1 . Let $\gamma \in K_1^\times$ denote the Teichmüller lift of $\bar{\gamma}$, and set

$$E = K_1((\gamma\varpi_K)^{1/2r}),$$

which is a totally (tamely) ramified cyclic Galois extension of K_1 of degree $2r$ since $2r|(q^2 - 1)$. Let $\mathcal{O}_E$ denote the ring of integers of E with maximal ideal $\mathfrak{m}_E$. Let $W_E \subset W_{K_1}$ denote the corresponding Weil groups, and write $\mathrm{ab}_E : W_E \rightarrow W_E^{\mathrm{ab}}$ for the Abelianization map. Let

$$\mathrm{Art}_E : E^\times \xrightarrow{\sim} W_E^{\mathrm{ab}}$$

be the local Artin map, normalized so that uniformizers are mapped to geometric Frobenius classes.

Let τ be a generator of $\mathrm{Gal}(E/K_1) \cong \mathbb{Z}/2r\mathbb{Z}$ and let ϖ_E be a uniformizer of E such that $\tau(\varpi_E) = \zeta\varpi_E$ for some $2r$ -th root of unity $\zeta \in K_1^\times$. Let $\phi \in \mathrm{Gal}(E/K)$ be lift of $\mathfrak{c} \in \mathrm{Gal}(K_1/K)$. By considering the action of ϕ we may change the lift so that $\phi(\gamma) = \gamma^q$ and $\phi(\varpi_E) = -\varpi_E$. In particular, $\phi^2 = 1$.

Recall the group decomposition

$$E^\times = \langle \varpi_E \rangle \times \kappa_1^\times \times U_E^1, \quad U_E^1 = 1 + \mathfrak{m}_E,$$

where $\kappa_1^\times$ embeds into $K_1^\times$ via the Teichmüller lift $[-] : \kappa_1^\times \rightarrow K_1^\times$. Since $p > 2$, the p -adic logarithm

$$\log : U_E^1 \rightarrow \mathfrak{m}_E : 1 + x \mapsto \sum_{k \in \mathbb{Z}_+} \frac{(-x)^{k+1}}{k}$$

is a continuous group homomorphism and is $\mathrm{Gal}(E/K)$ -equivariant. We extend $\log$ to a map $E^\times \rightarrow \mathfrak{m}_E$ by setting $\log(\varpi_E) = 0$ and $\log|_{\kappa_1^\times} = 0$.

Let e_K denote the ramification index of K , and set $k_0 = \lfloor \frac{2re_K}{p-1} \rfloor + 1$. Fix a positive integer $m > k_0$ to be determined later. Define, for $x \in E$,

$$\Psi_E := \iota_\ell e^{2\pi i \cdot \mathrm{tr}_{E/\mathbb{Q}_p}(x)/p^{m+[K:\mathbb{Q}_p]}},$$

which is an additive character of E of conductor at most $-(2re_K(m-1) + 1)$. Let $\chi : E^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$ denote the character

$$\chi(x) := \Psi_E(\varpi_E \log x), \quad x \in E^\times.$$

When m is sufficiently large, $\chi^\phi = \chi^{-1}$ and $\chi^\sigma \neq \chi$ for every nontrivial element $\sigma \in \mathrm{Gal}(E/K_1)$. Here we use that $\mathfrak{m}_E^{k_0} \subset \log(E^\times)$. Set

$$\xi := \chi \circ \mathrm{Art}_E^{-1} \circ \mathrm{ab}_E : W_E \rightarrow \overline{\mathbb{Q}_\ell}^\times.$$

Let

$$(\rho, V) := \mathrm{Ind}_{W_E}^{W_{K_1}} \xi$$

denote the induced representation of W_{K_1} of dimension $2r$. It follows from Mackey's theory that ρ is absolutely irreducible. Fix an element $s \in W_K$ lifting $\phi \in \mathrm{Gal}(E/K)$, and define a pairing on V given by

$$\langle f, g \rangle := \sum_{[x] \in W_E \backslash W_{K_1}} f(x)g(sxs^{-1}).$$

Here for each $[x] \in W_E \backslash W_K$, $x \in W_K$ is a lift of $[x]$. Note that this is well-defined because replacing x by hx gives

$$f(hx)g(shxs^{-1}) = \xi(h)f(x)g(shs^{-1}(sxs^{-1})) = \xi(h)\xi^\phi(h)f(x)g(sxs^{-1}) = f(x)g(sxs^{-1}).$$

This pairing is clearly nondegenerate. We check (A.1):

$$\langle \rho(\tau)f, \rho(s\tau s^{-1})g \rangle = \sum_{[x] \in W_E \backslash W_{K_1}} f(x\tau)g(sx\tau s^{-1}) = \sum_{[x] \in W_E \backslash W_{K_1}} f(x)g(sxs^{-1}).$$

$$\langle g, f \rangle = \sum_{[x] \in W_E \setminus W_{K_1}} g(x) f(sxs^{-1}) = \sum_{[x] \in W_E \setminus W_{K_1}} f(x) g(s^{-1}xs) = \xi(s^2)^{-1} \langle f, \rho(s^2)g \rangle.$$

Here we use that conjugation by s permutes left W_E -cosets. We claim that $\xi(s^2) = 1$. In fact, this claim is independent of the lift s chosen, because for any other lift $s' = hs$ with $h \in W_E$,

$$\xi((s')^2) = \xi(h)\xi(shs^{-1})\xi(s^2) = \xi(h)\xi^\phi(h)\xi(s^2) = \xi(s^2).$$

To prove the claim, we let H denote the subgroup of elements of W_K whose images in $\text{Gal}(E/K)$ lie in $\langle \phi \rangle$. Then there is an exact sequence

$$1 \rightarrow \xi(W_E) \rightarrow H / \ker(\xi) \rightarrow \langle \phi \rangle \rightarrow 1.$$

Note that $\xi(W_E)$ is a finite p -group. So it follows from the Schur–Zassenhaus theorem that there exists a lift $s \in W_K$ of ϕ satisfying $s^2 \in \ker(\xi)$. In particular, $\xi(s^2) = 1$, and (A.1) is proved.

We check that ρ is residually absolutely irreducible: As ρ factors through $W_K / \ker(\xi)$, which is a finite group with order dividing $2rq^M$ for some positive integer M . Thus $\overline{\mathbb{F}_\ell}[\text{Im}(\overline{\rho})]$ is a semisimple algebra, because ℓ is coprime to $2pr$. Thus the same Mackey theory argument implies that ρ is residually absolutely irreducible.

The lemma is proved. $\square$

Definition A.1.2. Let ℓ be a rational prime distinct from p , with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. Let B be a finite subset of $\overline{\mathbb{F}_\ell}$. A B -avoiding good representation (with respect to ι_ℓ) is a representation Π of $\text{GL}_{2r}(K_1)$ such that there exists some lift $F \in \text{Gal}_K$ of the arithmetic Frobenius element satisfying

- there is a partition $n = \sum_{i=1}^k n_i$ such that Π is an isobaric sum of distinct representations Π_i where each Π_i is a supercuspidal representation of $\text{GL}_{n_i}(K_1)$;
- for each $1 \leq i \leq k$, if we write $\Pi_i^\theta := (\Pi_i^s)^\vee$, where Π_i^s is the conjugate of Π_i by an element $s \in W_K$ which maps to $\mathbf{c} \in \text{Gal}(K_1/K)$, then there is an isomorphism $f_i : \Pi_i^\theta \xrightarrow{\sim} \Pi_i$ satisfying $(f_i^\vee)^s = f_i$;
- the Galois representation $\iota_\ell \text{rec}_{2r}(\Pi) : W_{K_1} \rightarrow \text{GL}_{2r}(\overline{\mathbb{Q}_\ell})$ attached to Π via local Langlands maps F^2 to an element with generalized eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ in which α_i is an ℓ -adic unit with residue not in B for every $1 \leq i \leq 2r$.

Suppose Π_0 is a constituent of an unramified principal series of $\text{GL}_{2r}(K_1)$ with Satake parameter $\alpha(\Pi_0) = \{\beta_1, \dots, \beta_{2r}\}$. If $\iota_\ell(\beta_i)$ is an ℓ -adic unit for every $1 \leq i \leq r$, then we say a representation Π of $\text{GL}_{2r}(K_1)$ is Π_0 -avoiding (with respect to ι_ℓ) if it is $B := \{-q, q\iota_\ell(\beta_1), \dots, q\iota_\ell(\beta_{2r})\} \pmod{\ell}$ -avoiding.

For a given finite subset $B \subset \overline{\mathbb{F}_\ell}$, constructing B -avoiding good representations is more complicated than we expected. In fact, we do not know how to construct supercuspidal B -avoiding good representations. Nonetheless, we have the following result which is enough for our purpose.

Lemma A.1.3. Let ℓ be a rational prime distinct from p , with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. Suppose ℓ is coprime to $2pr$. For any finite subset $B \subset \overline{\mathbb{F}_\ell}$, a B -avoiding good representation exists. We can further ensure that the generalized eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ as defined in Definition A.1.2 satisfy

$$q^2 \notin \{\alpha_i \alpha_j^{-1} | 1 \leq i \neq j \leq 2r\} \cup \{\alpha_i | 1 \leq i \leq 2r\} \subset \overline{\mathbb{F}_\ell}.$$

Proof. By local Langlands correspondence for $\text{GL}_{2r}(K_1)$, it suffices to construct

- a $2r$ -dimensional representation (ρ, V) of W_{K_1} with $\overline{\mathbb{Q}_\ell}$ -coefficients that is a sum of r distinct 2-dimensional irreducible representations,
- a lift $F \in W_K$ of the arithmetic Frobenius element,
- a lift $s \in W_K$ of $\mathbf{c} \in \text{Gal}(K_1/K)$, and
- a nondegenerate pairing $\langle -, - \rangle : V \times V \rightarrow \overline{\mathbb{Q}_\ell}$

satisfying

$$(A.2) \quad \begin{cases} \langle \rho(\tau)f, \rho(s\tau s^{-1})g \rangle = \langle f, g \rangle \\ \langle g, f \rangle = \langle f, \rho(s^2)g \rangle \end{cases}$$

for all $\tau \in W_{K_1}$ and $f, g \in V$; and

(2) The eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ of $\rho(F^2)$ are ℓ -adic units with residues not in B , and

$$q^2 \notin \{\alpha_i \alpha_j^{-1} | 1 \leq i \neq j \leq 2r\} \cup \{\alpha_i | 1 \leq i \leq 2r\} \subset \overline{\mathbb{F}_\ell}.$$

Let R/K be a quadratic ramified extension and let $E = RK_1$. Let $\mathcal{O}_E$ denote the ring of integers of E with maximal ideal $\mathfrak{m}_E$. Let τ denote the nontrivial element of $\text{Gal}(R/K) \cong \mathbb{Z}/2\mathbb{Z}$. Then there is a natural identification

$$\text{Gal}(E/K) = \langle \tau \rangle \times \langle c \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Let $W_E \subset W_{K_1} \subset W_K$ denote the corresponding Weil groups, and let $\text{ab}_E : W_E \rightarrow W_E^{\text{ab}}$ denote the Abelianization map. Let the Artin map

$$\text{Art}_E : E^\times \xrightarrow{\sim} W_E^{\text{ab}}$$

be normalized so that uniformizers are mapped to geometric Frobenius classes. Choose a uniformizer ϖ_R of R with $\tau(\varpi_R) = -\varpi_R$.

Recall the group decomposition

$$E^\times = \langle \varpi_R \rangle \times \kappa_1^\times \times U_E^1, \quad U_E^1 = 1 + \mathfrak{m}_E,$$

where $\kappa_1^\times$ embeds into $K_1^\times$ via the Teichmüller lift $[-] : \kappa_1^\times \rightarrow K_1^\times$. Since $p > 2$, the p -adic logarithm

$$\log : U_E^1 \rightarrow \mathfrak{m}_E : 1 + x \mapsto \sum_{k \in \mathbb{Z}_+} \frac{(-x)^{k+1}}{k}$$

is a continuous group homomorphism and is $\text{Gal}(E/K)$ -equivariant. We extend $\log$ to a map $E^\times \rightarrow \mathfrak{m}_E$ by setting $\log(\varpi_R) = 0$ and $\log|_{\kappa_1^\times} = 0$.

Let d_K denote the different exponent of K , so the different ideal $\mathfrak{d}_K$ of K over $\mathbb{Q}_p$ satisfies $\mathfrak{d}_K = \mathfrak{m}_K^{d_K}$. Let e_K denote the ramification index of K , and set $k_0 = \lfloor \frac{2e_K}{p-1} \rfloor + 1$. Fix a positive integer $m > k_0$ to be determined later. Let

$$\Psi_{\mathbb{Q}_p} : \mathbb{Q}_p \rightarrow \overline{\mathbb{Q}_\ell}^\times : x \mapsto \iota_\ell e^{2\pi i x}$$

denote the standard additive character of $\mathbb{Q}_p$ of conductor 0, and set

$$\Psi_K := \Psi_{\mathbb{Q}_p} \left(\text{tr}_{K/\mathbb{Q}_p}(\varpi_K^{-d_K-m} x) \right), \quad \Psi_E := \Psi_K \circ \text{tr}_{E/K}$$

Then Ψ_E is an additive character of E of conductor $1 - 2m$.

For each $1 \leq i \leq r$, let $\chi_i : E^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$ denote the character given by

$$\chi_i(x) := \Psi_E(p^{1-i} \varpi_R \log x), \quad x \in E^\times.$$

Set $\phi = \tau c \in \text{Gal}(E/K)$. Then $\chi_i^\phi = \chi_i^\tau = \chi_i^{-1} \neq \chi_i$ for every $1 \leq i \leq r$. It is clear that $\chi_i \neq \chi_j^\pm$ for $1 \leq i < j \leq r$. Here we use that $\mathfrak{m}_E^{k_0} \subset \log(E^\times)$. Set

$$\xi_i := \chi_i \circ \text{Art}_E^{-1} \circ \text{ab}_E : W_E \rightarrow \overline{\mathbb{Q}_\ell}^\times, \quad 1 \leq i \leq r.$$

For each $1 \leq i \leq r$, let

$$(\rho_i, V_i) := \text{Ind}_{W_E}^{W_{K_1}} \xi_i$$

denote the induced representation of W_K of dimension $2r$. And we define

$$(\rho, V) := \bigoplus_{i=1}^r (\rho_i, V_i).$$

It follows from Mackey's theory that ρ is a direct sum of r distinct 2-dimensional absolutely irreducible representations. Fix any element $s \in W_K$ lifting $\phi \in \text{Gal}(E/K)$, and define a pairing on V_i given by

$$\langle (f_1, \dots, f_r), (g_1, \dots, g_r) \rangle := \sum_{i=1}^r \sum_{[x] \in W_E \setminus W_{K_1}} f_i(x) g_i(sxs^{-1}).$$

Here for each $[x] \in W_E \setminus W_K$, $x \in W_K$ is a lift of $[x]$. The same argument as in the proof of Lemma A.1.1 shows that (A.2) holds.

We compute the Frobenius eigenvalues of the residual representation of ρ . Fix a lift $F_0 \in W_K$ of $\phi \in \text{Gal}(E/K)$. For each $t \in I_E$, $F = tF_0$ is also a lift of $\mathfrak{c} \in \text{Gal}(E/K)$. The characteristic polynomial of $\bar{\rho}(F^2)$ is

$$\chi_{\bar{\rho}(F^2)}(X) = \prod_{i=1}^r (X - \bar{\xi}_i(F^2))(X - \bar{\xi}_i(F^2)^{-1}).$$

For each $1 \leq i \leq r$, note that

$$\begin{aligned} \xi_i(F^2) &= \xi_i(F_0^2) \Psi_E(p^{1-i} \varpi_R \text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))) \\ &= \xi(F_0^2) \Psi_{\mathbb{Q}_p} \left(\text{tr}_{K/\mathbb{Q}_p} \varpi_K^{-d_K-m} \text{tr}_{R/K} (2p^{1-i} \varpi_R \text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))) \right). \end{aligned}$$

When t varies, $\log \text{Art}_E^{-1} \text{ab}_E(t)$ ranges over all elements in $\mathfrak{m}_E^{k_0}$. Since E/R is unramified, $\text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(t))$ ranges over all elements in $\mathfrak{m}_R^{k_0}$, and

$$\text{tr}_{R/K} (2p^{1-i} \varpi_R \text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i)))$$

ranges over all elements in $\mathfrak{m}_K^{k_0}$. Thus when t varies,

$$\Psi_{\mathbb{Q}_p} \left(\text{tr}_{K/\mathbb{Q}_p} \varpi_K^{-d_K-m} \text{tr}_{R/K} (2p^{1-i} \varpi_R \text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))) \right)$$

can be every $p^{\lfloor \frac{m-k_0}{e_K} \rfloor}$ -th roots of unity in $\overline{\mathbb{Q}_\ell}$. Thus, when s is large, it is clear that we can take some $t \in I_E$ such that $\bar{\xi}_i(F^2)$ is not contained in the set

$$\{b^{\pm 1} | b \in B\} \cup \{q^{\pm 1}, q^{\pm 2}\} \subset \overline{\mathbb{F}_\ell}$$

for every $1 \leq i \leq r$. As a result, $\bar{\rho}(F^2)$ has no eigenvalues in B .

Similarly, when s is large, we can further assume that $t \in I_E$ is chosen so that each of the elements

$$\bar{\xi}_i(F^2) \bar{\xi}_j(F^2) = \xi_i(F_0^2) \Psi_E((p^{1-i} + p^{1-j}) \varpi_R \text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))), \quad 1 \leq i < j \leq r$$

and

$$\bar{\xi}_i(F^2) \bar{\xi}_j^{-1}(F^2) = \xi_i(F_0^2) \Psi_E((p^{1-i} - p^{1-j}) \varpi_R \text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))), \quad 1 \leq i \leq r$$

is not contained in $\{q^{\pm 2}\} \subset \overline{\mathbb{F}_\ell}$.

The desired properties of ρ are all proved. $\square$

A.2. Special self-dual local Galois representations. Let p be an odd rational prime, and let K be a finite extension of $\mathbb{Q}_p$. Let κ denote the residue field of K , of cardinality q . Let $\mathcal{O}_K$ denote the ring of integers of K with maximal ideal $\mathfrak{m}_K$. Fix a uniformizer ϖ_K of K .

Definition A.2.1. Let ℓ be a rational prime distinct from p , with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. For a finite subset $B \subset \overline{\mathbb{F}_\ell}$, a supercuspidal B -avoiding good representation (with respect to ι_ℓ) is a supercuspidal representation Π of $\text{GL}_{2r}(K)$ such that there exists some lift $F \in \text{Gal}_K$ of the arithmetic Frobenius element satisfying

- there is an isomorphism $f : \Pi^\vee \xrightarrow{\sim} \Pi$ satisfying $f^\vee = f$.
- the Galois representation $\iota_\ell \text{rec}_{2r}(\Pi) : W_K \rightarrow \text{GL}_{2r}(\overline{\mathbb{Q}_\ell})$ attached to Π via local Langlands maps F to an element with generalized eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ in which α_i is an ℓ -adic unit with residue not in B for every $1 \leq i \leq 2r$.

If Π_0 is a constituent of an unramified principal series of $\text{GL}_{2r}(K)$ with Satake parameter $\alpha(\Pi_0) = \{\beta_1, \dots, \beta_{2r}\}$, then we say a representation Π of $\text{GL}_{2r}(K)$ is Π_0 -avoiding if it is B -avoiding for

$$B := \{\pm 1, \pm q^{\pm 1}, \dots, \pm q^{\pm 4r}\} \cup \{q^{1/2} \beta_1, \dots, q^{1/2} \beta_{2r}\} \subset \overline{\mathbb{F}_\ell}.$$

Lemma A.2.2. Let ℓ be a rational prime with an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ satisfying $\ell \nmid 2pr$. For any finite subset $B \subset \overline{\mathbb{F}_\ell}$, a supercuspidal B -avoiding good representation Π exists. Moreover, we can make sure that $\iota_\ell \text{rec}_{2r}(\Pi)$ is absolutely residually irreducible. If $q^{2r} - 1$ is not divisible by ℓ , we can further make sure that the generalized eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ as defined in Definition A.2.1 satisfy

$$q \notin \{\pm \alpha_i \alpha_j^{-1} | 1 \leq i \neq j \leq 2r\} \cup \{\pm \alpha_i | 1 \leq i \leq 2r\} \subset \overline{\mathbb{F}_\ell}.$$

Proof. It suffices to show that we can find a residually absolutely irreducible $2r$ -dimensional representation (ρ, V) of W_K with $\overline{\mathbb{Q}_\ell}$ -coefficients and a lift $F \in \text{Gal}_K$ of the arithmetic Frobenius element satisfying

- there exists a W_K -invariant nondegenerate $\overline{\mathbb{Q}_\ell}$ -valued symmetric pairing on V ;
- the eigenvalues $\{\alpha_1, \dots, \alpha_{2r}\}$ of $\rho(F)$ are ℓ -adic units with residues not in B ; and
-

$$q \notin \{\pm \alpha_i \alpha_j^{-1} | 1 \leq i \neq j \leq 2r\} \cup \{\pm \alpha_i | 1 \leq i \leq 2r\} \subset \overline{\mathbb{F}_\ell}.$$

holds if $q^{2r} - 1$ is not divisible by ℓ .

Choose U/K unramified of degree r with Frobenius class σ , and R/K ramified quadratic with Galois group $\text{Gal}(R/K) = \{1, \tau\}$. Set $E = UR$. Let $\mathcal{O}_E$ denote the ring of integers of E with maximal ideal $\mathfrak{m}_E$. Then there is a natural identification

$$\text{Gal}(E/K) = \langle \phi \rangle \times \langle \tau \rangle \cong \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Let $W_E \subset W_K$ be the corresponding Weil groups, and let $\text{ab}_E : W_E \rightarrow W_E^{\text{ab}}$ denote the Abelianization map. Let the Artin map

$$\text{Art}_E : E^\times \xrightarrow{\sim} W_E^{\text{ab}}$$

be normalized so that uniformizers are mapped to geometric Frobenius classes.

Let κ_U denote the residue field of U . Choose a uniformizer ϖ_R of R satisfying $\tau(\varpi_R) = -\varpi_R$. Recall the decomposition

$$E^\times = \langle \varpi_R \rangle \times \kappa_U^\times \times U_E^1, \quad U_E^1 = 1 + \mathfrak{m}_E,$$

where $\kappa_U^\times$ embeds into $K^\times$ via the Teichmüller lift $[-] : \kappa_U^\times \rightarrow K^\times$. Since $p > 2$, the p -adic logarithm

$$\log : U_E^1 \rightarrow \mathfrak{m}_E : 1 + x \mapsto \sum_{k \in \mathbb{Z}_+} \frac{(-x)^{k+1}}{k}$$

is a continuous group homomorphism and is $\text{Gal}(E/K)$ -equivariant. We extend $\log$ to a map $E^\times \rightarrow \mathfrak{m}_E$ by setting $\log(\varpi_R) = 0$ and $\log|_{\kappa_U^\times} = 0$.

Let d_K denote the different exponent of K , so the different ideal $\mathfrak{d}_K$ of K over $\mathbb{Q}_p$ satisfies $\mathfrak{d}_K = \mathfrak{m}_K^{d_K}$. Let e_K denote the ramification index of K , and set $k_0 = \lfloor \frac{2e_K}{p-1} \rfloor + 1$. Fix a positive integer $s > k_0$ to be determined later. Let

$$\Psi_{\mathbb{Q}_p} : \mathbb{Q}_p \rightarrow \overline{\mathbb{Q}_\ell}^\times : x \mapsto \iota_\ell e^{2\pi i x}$$

denote the standard additive character of $\mathbb{Q}_p$ of conductor 0, and set

$$\Psi_K := \Psi_{\mathbb{Q}_p} \left(\text{tr}_{K/\mathbb{Q}_p}(\varpi_K^{-d_K-s} x) \right), \quad \Psi_E := \Psi_K \circ \text{tr}_{E/K}$$

Then Ψ_E is an additive character of E of conductor $1 - 2s$.

Take an element $\bar{\gamma} \in \kappa_U^\times$ satisfying

- (1) $\bar{\gamma}^{q^i} \neq \pm \bar{\gamma}$ for every $1 \leq i \leq r-1$, and
- (2) $\text{tr}_{\kappa_U/\kappa}(\bar{\gamma}) \neq 0$.

Such an element $\bar{\gamma}$ exists by normal basis theorem. Indeed, we can take $\bar{\gamma}$ such that

$$\{\sigma^{q^i}(\bar{\gamma}) | 0 \leq i \leq r-1\}$$

is a κ -basis of κ_U . Set $\alpha = \varpi_R[\bar{\gamma}]$. Then

- (1) $\tau(\alpha) = -\alpha$, and
- (2) $\sigma(\alpha) - \alpha \in \mathfrak{m}_E \setminus \mathfrak{m}_E^2$ for every nontrivial element $\sigma \in \text{Gal}(E/K)$.

Let $\chi : E^\times \rightarrow \overline{\mathbb{Q}_\ell}^\times$ denote the character given by

$$\chi(x) := \Psi_E(\alpha \log x), \quad x \in E^\times.$$

Then $\chi^\tau = \chi^{-1}$ and $\chi^\sigma \neq \chi$ for every nontrivial element $\sigma \in \text{Gal}(E/K)$. Here we use that $\mathfrak{m}_E^{k_0} \subset \log(E^\times)$. Set

$$\xi := \chi \circ \text{Art}_E^{-1} \circ \text{ab}_E : W_E \rightarrow \overline{\mathbb{Q}_\ell}^\times.$$

Let

$$(\rho, V) := \text{Ind}_{W_E}^{W_K} \xi$$

be the induced representation of W_K of dimension $2r$. It follows from Mackey's theory that ρ is absolutely irreducible. Fix any element $y \in W_K$ lifting $\tau \in \text{Gal}(E/K)$, and define a pairing on V given by

$$\langle f, g \rangle := \sum_{[x] \in W_E \setminus W_K} f(x)g(y^{-1}x).$$

Here for each $[x] \in W_E \setminus W_K$, $x \in W_K$ is a lift of $[x]$. Note that this is well-defined because replacing x by hx gives

$$f(hx)g(y^{-1}hx) = \xi(h)f(x)g(y^{-1}hy(y^{-1}x)) = \xi(h)\xi^\tau(h)f(x)g(y^{-1}x) = f(x)g(y^{-1}x).$$

This pairing is clearly W_K -invariant and nondegenerate. We check that it is symmetric:

$$\begin{aligned} \langle f, g \rangle &= \sum_{[x] \in W_E \setminus W_K} f(x)g(y^{-1}x) \\ &= \sum_{[x] \in W_E \setminus W_K} f(yxy^{-1})g(xy^{-1}) \\ &= \sum_{[x] \in W_E \setminus W_K} f(yx)g(x) \\ &= \xi(y^2) \sum_{[x] \in W_E \setminus W_K} g(x)f(y^{-1}x) \\ &= \xi(y^2) \langle g, f \rangle \end{aligned}$$

Here we use that conjugation by y permutes left W_E -cosets. We claim that $\xi(y^2) = 1$. In fact, this claim is independent of the lift y chosen, because for any other lift $y' = hy$ with $h \in W_E$,

$$\xi((y')^2) = \xi(h)\xi(yhy^{-1})\xi(y^2) = \xi(h)\xi^\tau(h)\xi(y^2) = \xi(y^2).$$

To prove the claim, we let H denote the subgroup of elements of W_K whose images in $\text{Gal}(E/K)$ lie in $\langle \tau \rangle$. Then there is an exact sequence

$$1 \rightarrow \xi(W_E) \rightarrow H / \ker(\xi) \rightarrow \langle \tau \rangle \rightarrow 1.$$

Note that $\xi(W_E)$ is a finite p -group. So it follows from the Schur–Zassenhaus theorem that there exists a lift $y \in W_K$ of τ satisfying $y^2 \in \ker(\xi)$. In particular, $\xi(y^2) = 1$, and the form $\langle -, - \rangle$ is symmetric.

We check that ρ is residually absolutely irreducible: As ρ factors through $W_K / \ker(\xi)$, which is a finite group with order dividing $2rq^{r(2s-3)}$. Thus $\overline{\mathbb{F}}_\ell[\text{Im}(\bar{\rho})]$ is a semisimple algebra, because ℓ is coprime to $2pr$. Thus the same Mackey theory argument implies that ρ is residually absolutely irreducible.

We compute the Frobenius eigenvalues of the residual representation of ρ . Fix a lift $F_0 \in W_K$ of $\phi \in \text{Gal}(E/K)$. For each $t \in I_E$, $F = tF_0$ is also lift of $\langle \phi \rangle \in \text{Gal}(E/K)$. The characteristic polynomial of $\bar{\rho}(F)$ is

$$\chi_{\bar{\rho}(F)}(X) = (X^r - \bar{\xi}(F^r))(X^r - \bar{\xi}(F^r)^{-1}).$$

Note that

$$\begin{aligned} \xi(F^r) &= \xi(F_0^r)\Psi_E(\alpha \text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))) \\ &= \xi(F_0^r)\Psi_{\mathbb{Q}_p}\left(\text{tr}_{K/\mathbb{Q}_p}\varpi_K^{-d_K-s}\text{tr}_{R/K}\left(\text{tr}_{E/R}(\alpha)\text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))\right)\right). \end{aligned}$$

It follows from the choice of γ that

$$\text{tr}_{E/R}(\alpha) = \varpi_R \text{tr}_{E/R} \gamma \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2.$$

When t varies, $\log \text{Art}_E^{-1} \text{ab}_E(t)$ ranges over all elements in $\mathfrak{m}_E^{k_0}$. Since E/R is unramified and $\text{tr}_{E/R}(\alpha) \in \mathfrak{m}_R \setminus \mathfrak{m}_R^2$, $\text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(t))$ ranges over all elements in $\mathfrak{m}_R^{k_0}$, and

$$\text{tr}_{R/K}\left(\text{tr}_{E/R}(\alpha)\text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))\right)$$

ranges over all elements in $\mathfrak{m}_K^{k_0}$. Thus when t varies,

$$\Psi_{\mathbb{Q}_p}\left(\text{tr}_{K/\mathbb{Q}_p}\varpi_K^{-d_K-s}\text{tr}_{R/K}\left(\text{tr}_{E/R}(\alpha)\text{tr}_{E/R}(\log \text{Art}_E^{-1} \text{ab}_E(i))\right)\right)$$

can be every $p^{\lfloor \frac{s-k_0}{e_K} \rfloor}$ -th roots of unity in $\overline{\mathbb{Q}_\ell}$. When s is large, it is clear that we can take some $t \in I_E$ such that $\bar{\xi}(F^r)$ is not contained in the set

$$\{b^{\pm r} | b \in B\} \cup \{(\pm q)^{\pm r}\} \subset \overline{\mathbb{F}_\ell}.$$

As a result, $\bar{\rho}(F)$ has no eigenvalues in B . Moreover, if $q^{2r} - 1$ is not divisible by ℓ , it is clear that

$$q \notin \{\pm \alpha_i \alpha_j^{-1} | 1 \leq i \neq j \leq 2r\} \cup \{\pm \alpha_i | 1 \leq i \leq 2r\} \subset \overline{\mathbb{F}_\ell}$$

is satisfied.

The desired properties of ρ are all proved. □

REFERENCES

- [AB98] J. Adams and D. Barbasch, *Genuine representations of the metaplectic group*, *Compositio Math.* **113** (1998), no. 1, 23–66. MR1638210
- [AC89] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Princeton University Press, 1989.
- [AG17] H. Atoke and W. T. Gan, *On the local Langlands correspondence and Arthur conjecture for even orthogonal groups*, *Represent. Theory* **21** (2017), 354–415. MR3708200
- [AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, *Lie Groups: History, Frontiers and Applications*, vol. Vol. IV, Math Sci Press, Brookline, MA, 1975. MR457437
- [And04] Y. André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004. MR2115000
- [Art13] J. Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups. MR3135650
- [BC83] A. Borel and W. Casselman, *L^2 -cohomology of locally symmetric manifolds of finite volume*, *Duke Math. J.* **50** (1983), no. 3, 625–647. MR714821
- [BD05] M. Bertolini and H. Darmon, *Iwasawa’s main conjecture for elliptic curves over anticyclotomic $\mathbb{Z}_p$ -extensions*, *Ann. of Math. (2)* **162** (2005), no. 1, 1–64. MR2178960
- [BFK⁺23] V. Blomer, É. Fouvry, E. Kowalski, P. Michel, D. Milićević, and W. Sawin, *The second moment theory of families of L -functions—the case of twisted Hecke L -functions*, *Mem. Amer. Math. Soc.* **282** (2023), no. 1394, v+148. MR4539366
- [BG14] K. Buzzard and T. Gee, *The conjectural connections between automorphic representations and Galois representations*, *Automorphic forms and Galois representations*. Vol. 1, 2014, pp. 135–187. MR3444225
- [BK90] S. Bloch and K. Kato, *L -functions and Tamagawa numbers of motives*, *The Grothendieck Festschrift*, Vol. I, 1990, pp. 333–400. MR1086888
- [BP14] R. Beuzart-Plessis, *Expression d’un facteur epsilon de paire par une formule intégrale*, *Canad. J. Math.* **66** (2014), no. 5, 993–1049. MR3251763
- [BP15] R. Beuzart-Plessis, *Endoscopie et conjecture locale raffinée de Gan-Gross-Prasad pour les groupes unitaires*, *Compos. Math.* **151** (2015), no. 7, 1309–1371. MR3371496
- [BP16] R. Beuzart-Plessis, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes unitaires*, *Mém. Soc. Math. Fr. (N.S.)* **149** (2016), vii+191. MR3676153
- [BPCZ22] R. Beuzart-Plessis, P.-H. Chaudouard, and M. Zydor, *The global Gan-Gross-Prasad conjecture for unitary groups: the endoscopic case*, *Publ. Math. Inst. Hautes Études Sci.* **135** (2022), 183–336. MR4426741
- [BPLZZ21] R. Beuzart-Plessis, Y. Liu, W. Zhang, and X. Zhu, *Isolation of cuspidal spectrum, with application to the Gan-Gross-Prasad conjecture*, *Ann. of Math. (2)* **194** (2021), no. 2, 519–584. MR4298750
- [BS91] M. Burger and P. Sarnak, *Ramanujan duals. II*, *Invent. Math.* **106** (1991), no. 1, 1–11. MR1123369
- [Car12] A. Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, *Duke Math. J.* **161** (2012), no. 12, 2311–2413. MR2972460
- [Car14] ———, *Monodromy and local-global compatibility for $l = p$* , *Algebra Number Theory* **8** (2014), no. 7, 1597–1646. MR3272276
- [CH13] G. Chenevier and M. Harris, *Construction of automorphic Galois representations, II*, *Camb. J. Math.* **1** (2013), no. 1, 53–73. MR3272052
- [Clo90] L. Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, *Automorphic forms, Shimura varieties, and L -functions*, Vol. I (Ann Arbor, MI, 1988), 1990, pp. 77–159. MR1044819
- [CZ21] R. Chen and J. Zou, *Local Langlands correspondence for even orthogonal groups via theta lifts*, *Selecta Math. (N.S.)* **27** (2021), no. 5, Paper No. 88, 71. MR4308934
- [CZ24] ———, *Arthur’s multiplicity formula for even orthogonal and unitary groups*, Preprint (2024), available at [arXiv:2103.07956](https://arxiv.org/abs/2103.07956).
- [Del79] P. Deligne, *Valeurs de fonctions L et périodes d’intégrales*, *Automorphic forms, representations and L -functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, 1979, pp. 313–346. With an appendix by N. Koblitz and A. Ogus. MR546622
- [Fal83] G. Faltings, *On the cohomology of locally symmetric Hermitian spaces*, *Paul Dubreil and Marie-Paule Malliavin algebra seminar, 35th year* (Paris, 1982), 1983, pp. 55–98. MR732471

- [FH95] S. Friedberg and J. Hoffstein, *Nonvanishing theorems for automorphic L -functions on $GL(2)$* , Ann. of Math. (2) **142** (1995), no. 2, 385–423. MR1343325
- [Fin21] J. Fintzen, *Types for tame p -adic groups*, Ann. of Math. (2) **193** (2021), no. 1, 303–346. MR4199732
- [FP23] N. Fakhruddin and V. Pilloni, *Hecke operators and the coherent cohomology of Shimura varieties*, J. Inst. Math. Jussieu **22** (2023), no. 1, 1–69. MR4556929
- [Gar05] P. Garrett, *Universality of holomorphic discrete series*, notes available at homepage of author (2005). https://www-users.cse.umn.edu/~garrett/m/v/holo_disc.pdf.
- [GGP12] W. T. Gan, B. H. Gross, and D. Prasad, *Symplectic local root numbers, central critical l -values, and restriction problems in the representation theory of classical groups*, Sur les conjectures de gross et prasad. i, 2012, pp. 1–109 (en). MR3202556
- [GI16] W. T. Gan and A. Ichino, *The Gross-Prasad conjecture and local theta correspondence*, Invent. Math. **206** (2016), no. 3, 705–799. MR3573972
- [GI18] ———, *The Shimura-Waldspurger correspondence for Mp_{2n}* , Ann. of Math. (2) **188** (2018), no. 3, 965–1016. MR3866889
- [GP92] B. H. Gross and D. Prasad, *On the decomposition of a representation of SO_n when restricted to SO_{n-1}* , Canad. J. Math. **44** (1992), no. 5, 974–1002. MR1186476
- [GP94] ———, *On irreducible representations of $SO_{2n+1} \times SO_{2m}$* , Canad. J. Math. **46** (1994), no. 5, 930–950. MR1295124
- [GRS93] S. Gelbart, J. Rogawski, and D. Soudry, *On periods of cusp forms and algebraic cycles for $U(3)$* , Israel J. Math. **83** (1993), no. 1-2, 213–252. MR1239723
- [GS12] W. T. Gan and G. Savin, *Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence*, Compos. Math. **148** (2012), no. 6, 1655–1694. MR2999299
- [GT16] W. T. Gan and S. Takeda, *A proof of the Howe duality conjecture*, J. Amer. Math. Soc. **29** (2016), no. 2, 473–493. MR3454380
- [GZ86] B. H. Gross and D. B. Zagier, *Heegner points and derivatives of L -series*, Invent. Math. **84** (1986), no. 2, 225–320. MR833192
- [Har07] M. Harris, *Cohomological automorphic forms on unitary groups. II. Period relations and values of L -functions*, Harmonic analysis, group representations, automorphic forms and invariant theory, 2007, pp. 89–149. MR2401812
- [HJL23] J. Hoffstein, J. Jung, and M. Lee, *Non-vanishing of symmetric cube L -functions*, J. Lond. Math. Soc. (2) **107** (2023), no. 1, 153–188. MR4535011
- [HKS96] M. Harris, S. S. Kudla, and W. J. Sweet, *Theta dichotomy for unitary groups*, J. Amer. Math. Soc. **9** (1996), no. 4, 941–1004. MR1327161
- [HL98] M. Harris and J.-S. Li, *A Lefschetz property for subvarieties of Shimura varieties*, J. Algebraic Geom. **7** (1998), no. 1, 77–122. MR1620690
- [How06] B. Howard, *Bipartite Euler systems*, J. Reine Angew. Math. **597** (2006), 1–25. MR2264314
- [How89a] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), no. 2, 539–570. MR986027
- [How89b] ———, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), no. 3, 535–552. MR985172
- [II10] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425. MR2585578
- [Ish24] H. Ishimoto, *The endoscopic classification of representations of non-quasi-split odd special orthogonal groups*, Int. Math. Res. Not. IMRN **2024** (2024), no. 14, 10939–11012. MR4776199
- [JR11] H. Jacquet and S. Rallis, *On the Gross-Prasad conjecture for unitary groups*, On certain L -functions, 2011, pp. 205–264. MR2767518
- [JS81] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. **103** (1981), no. 4, 777–815. MR623137
- [Kat04] K. Kato, *p -adic Hodge theory and values of zeta functions of modular forms*, 2004, pp. ix, 117–290. Cohomologies p -adiques et applications arithmétiques. III. MR2104361
- [KMSW14] T. Kaletha, A. Minguez, S. W. Shin, and P.-J. White, *Endoscopic classification of representations: Inner forms of unitary groups*, Preprint (2014), available at [arXiv:1409.3731](https://arxiv.org/abs/1409.3731).
- [Kol90] V. A. Kolyvagin, *Euler systems*, The Grothendieck Festschrift, Vol. II, 1990, pp. 435–483. MR1106906
- [KR05] S. S. Kudla and S. Rallis, *On first occurrence in the local theta correspondence*, Automorphic representations, L -functions and applications: progress and prospects, 2005, pp. 273–308. MR2192827
- [KR90] ———, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), no. 1, 25–45. MR1046171
- [KR94] ———, *A regularized Siegel-Weil formula: the first term identity*, Ann. of Math. (2) **140** (1994), no. 1, 1–80. MR1289491
- [KSZ21] M. Kisin, S. W. Shin, and Y. Zhu, *The stable trace formula for Shimura varieties of abelian type*, Preprint (2021), available at [arXiv:2110.05381](https://arxiv.org/abs/2110.05381).
- [Kud86] S. S. Kudla, *On the local theta-correspondence*, Invent. Math. **83** (1986), no. 2, 229–255. MR818351
- [Kud94] ———, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math. **87** (1994), no. 1-3, 361–401. MR1286835
- [Li90] J.-S. Li, *Theta lifting for unitary representations with nonzero cohomology*, Duke Math. J. **61** (1990), no. 3, 913–937. MR1084465
- [Liu16] Y. Liu, *Hirzebruch-Zagier cycles and twisted triple product Selmer groups*, Invent. Math. **205** (2016), no. 3, 693–780. MR3539925
- [Liu19] ———, *Bounding cubic-triple product Selmer groups of elliptic curves*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 5, 1411–1508. MR3941496

- [Liu21] ———, *Fourier-Jacobi cycles and arithmetic relative trace formula (with an appendix by Chao Li and Yihang Zhu)*, Camb. J. Math. **9** (2021), no. 1, 1–147. Appendix by Chao Li and Yihang Zhu. MR4325259
- [LL21] C. Li and Y. Liu, *Chow groups and L -derivatives of automorphic motives for unitary groups*, Ann. of Math. (2) **194** (2021), no. 3, 817–901. MR4334978
- [Lom15] D. Lombardo, *Bounds for Serre’s open image theorem for elliptic curves over number fields*, Algebra Number Theory **9** (2015), no. 10, 2347–2395. MR3437765
- [Lon06] M. Longo, *On the Birch and Swinnerton-Dyer conjecture for modular elliptic curves over totally real fields*, Annales de l’Institut Fourier **56** (2006), no. 3, 689–733 (en). MR2244227
- [Lon07] ———, *Euler systems obtained from congruences between Hilbert modular forms*, Rendiconti del Seminario Matematico della Università di Padova **118** (2007).
- [LTX24] Y. Liu, Y. Tian, and L. Xiao, *Iwasawa’s main conjecture for Rankin–Selberg motives in the anticyclotomic case* (2024), available at [arXiv:2406.00624](https://arxiv.org/abs/2406.00624).
- [LTX+22] Y. Liu, Y. Tian, L. Xiao, W. Zhang, and X. Zhu, *On the Beilinson–Bloch–Kato conjecture for Rankin–Selberg motives*, Invent. Math. **228** (2022), no. 1, 107–375. MR4392458
- [LTX+24] Y. F. Liu, Y. C. Tian, L. Xiao, W. Zhang, and X. W. Zhu, *Deformation of rigid conjugate self-dual Galois representations*, Acta Math. Sin. (Engl. Ser.) **40** (2024), no. 7, 1599–1644. MR4777059
- [LTX+25] Y. Liu, Y. Tian, L. Xiao, W. Zhang, and X. Zhu, *Survey on bounding Selmer groups for Rankin–Selberg motives* (2025), available at [arXiv:2509.16881](https://arxiv.org/abs/2509.16881).
- [LXZ25a] S. Leslie, J. Xiao, and W. Zhang, *Unitary Friedberg–Jacquet periods and their twists: Fundamental lemmas* (2025), available at [arXiv:2503.09500](https://arxiv.org/abs/2503.09500).
- [LXZ25b] ———, *Unitary Friedberg–Jacquet periods and their twists: Relative trace formulas* (2025), available at [arXiv:2503.09664](https://arxiv.org/abs/2503.09664).
- [Mín12] A. Mínguez, *The conservation relation for cuspidal representations*, Math. Ann. **352** (2012), no. 1, 179–188. MR2885581
- [Mok15] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248. MR3338302
- [MR18] C. Moeglin and D. Renard, *Sur les paquets d’Arthur des groupes classiques et unitaires non quasi-déployés*, Relative aspects in representation theory, Langlands functoriality and automorphic forms, 2018, pp. 341–361. MR3839702
- [MVW87] C. Moeglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p -adique*, Lecture Notes in Mathematics, vol. 1291, Springer-Verlag, Berlin, 1987. MR1041060
- [Nek00] J. Nekovář, *p -adic Abel–Jacobi maps and p -adic heights*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 2000, pp. 367–379. MR1738867
- [Nek12] ———, *Level raising and anticyclotomic Selmer groups for Hilbert modular forms of weight two*, Canad. J. Math. **64** (2012), no. 3, 588–668. MR2962318
- [NN16] J. Nekovář and W. Nizioł, *Syntomic cohomology and p -adic regulators for varieties over p -adic fields*, Algebra Number Theory **10** (2016), no. 8, 1695–1790. With appendices by Laurent Berger and Frédéric Déglise. MR3556797
- [NT21] J. Newton and J. A. Thorne, *Symmetric power functoriality for holomorphic modular forms, II*, Publ. Math. Inst. Hautes Études Sci. **134** (2021), 117–152. MR4349241
- [NT22] ———, *Symmetric power functoriality for Hilbert modular forms*, Preprint (2022), available at [arXiv:2212.03595](https://arxiv.org/abs/2212.03595).
- [Pen25] H. Peng, *Fargues–Scholze parameters and torsion vanishing for special orthogonal and unitary groups*, Preprint (2025), available at [arXiv:2503.04623](https://arxiv.org/abs/2503.04623).
- [Pen26] ———, *Higher Kolyagin theorems in the arithmetic Gross–Prasad setting*, Ph.D. thesis to appear (2026).
- [Pra07] D. Prasad, *Relating invariant linear form and local epsilon factors via global methods*, Duke Math. J. **138** (2007), no. 2, 233–261. With an appendix by Hiroshi Saito. MR2318284
- [Pra99] ———, *Some remarks on representations of a division algebra and of the Galois group of a local field*, J. Number Theory **74** (1999), no. 1, 73–97. MR1670568
- [PS79] I. I. Piatetski-Shapiro, *Multiplicity one theorems*, Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, 1979, pp. 209–212. MR546599
- [Rog92] J. D. Rogawski, *Analytic expression for the number of points mod p* , The zeta functions of Picard modular surfaces, 1992, pp. 65–109. MR1155227
- [Roh94] D. E. Rohrlich, *Elliptic curves and the Weil–Deligne group*, Elliptic curves and related topics, 1994, pp. 125–157. MR1260960
- [RR93] R. Ranga Rao, *On some explicit formulas in the theory of Weil representation*, Pacific J. Math. **157** (1993), no. 2, 335–371. MR1197062
- [Rub00] K. Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study. MR1749177
- [RY23] M. Radziwiłł and L. Yang, *Non-vanishing of twists of $GL_4(\mathbb{A}_n)$ L -functions* (2023), available at [arXiv:2304.09171](https://arxiv.org/abs/2304.09171).
- [Ser72] J.-P. Serre, *Propriétés galoisiennes des points d’ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), no. 4, 259–331. MR387283
- [ST14] S. W. Shin and N. Templier, *On fields of rationality for automorphic representations*, Compos. Math. **150** (2014), no. 12, 2003–2053. MR3292292
- [Swe25] N. Sweeting, *On the Bloch–Kato conjecture for some four-dimensional symplectic Galois representations*, Preprint (2025), available at [arXiv:2503.19226](https://arxiv.org/abs/2503.19226).
- [SZ15] B. Sun and C.-B. Zhu, *Conservation relations for local theta correspondence*, J. Amer. Math. Soc. **28** (2015), no. 4, 939–983. MR3369906

- [TY07] R. Taylor and T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493. MR2276777
- [Wal10] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad*, Compos. Math. **146** (2010), no. 5, 1180–1290. MR2684300
- [Wal12a] J.-L. Waldspurger, *Calcul d’une valeur d’un facteur ϵ par une formule intégrale*, 2012, pp. 1–102. Sur les conjectures de Gross et Prasad. II. MR3155344
- [Wal12b] ———, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes spéciaux orthogonaux*, 2012, pp. 103–165. Sur les conjectures de Gross et Prasad. II. MR3155345
- [Wal12c] ———, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad, 2e partie: extension aux représentations tempérées*, 2012, pp. 171–312. Sur les conjectures de Gross et Prasad. I. MR3202558
- [Wal90] J.-L. Waldspurger, *Démonstration d’une conjecture de dualité de Howe dans le cas p -adique, $p \neq 2$* , Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), 1990, pp. 267–324. MR1159105
- [Wan22] H. Wang, *Arithmetic level raising on triple product of Shimura curves and Gross-Schoen diagonal cycles, I: Ramified case*, Algebra Number Theory **16** (2022), no. 10, 2289–2338. MR4546490
- [Xue14] H. Xue, *The Gan-Gross-Prasad conjecture for $U(n) \times U(n)$* , Adv. Math. **262** (2014), 1130–1191. MR3228451
- [Yam14] S. Yamana, *L -functions and theta correspondence for classical groups*, Invent. Math. **196** (2014), no. 3, 651–732. MR3211043
- [YZ25] X. Yang and X. Zhu, *On the generic part of the cohomology of shimura varieties of abelian type* (2025), available at [arXiv:2505.04329](https://arxiv.org/abs/2505.04329).
- [Zan24] M. C. Zanarella, *First Explicit Reciprocity Law for Unitary Friedberg—Jacquet Periods*, ProQuest LLC, Ann Arbor, MI, 2024. Thesis (Ph.D.)—Massachusetts Institute of Technology. MR4890378
- [Zha01] S. Zhang, *Heights of Heegner points on Shimura curves*, Ann. of Math. (2) **153** (2001), no. 1, 27–147. MR1826411
- [Zha14] W. Zhang, *Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups*, Ann. of Math. (2) **180** (2014), no. 3, 971–1049. MR3245011

HAO PENG

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Email: hao_peng@mit.edu